

Weak complicial sets I. Basic homotopy theory

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Dedicated to Ross Street on the occasion of his 60th birthday

Abstract

This paper develops the foundations of a simplicial theory of weak ω -categories, which builds upon the insights originally expounded by Ross Street in his 1987 paper on oriented simplices. The resulting theory of *weak complicial sets* provides a common generalisation of the theories of (strict) ω -categories, Kan complexes and Joyal's *quasi-categories*. We generalise a number of results due to the current author with regard to complicial sets and strict ω -categories to provide an armoury of well behaved technical devices, such as joins and Gray tensor products, which will be used to study the weak ω -category theory of these structures in a series of companion papers. In particular, we establish their basic homotopy theory by constructing a Quillen model structure on the category of *stratified simplicial sets* whose fibrant objects are the weak complicial sets. As a simple corollary of this work we provide an independent construction of Joyal's model structure on simplicial sets for which the fibrant objects are the quasi-categories.

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1. Overview and history

The theory of *complicial sets* dates to the mid-1970s and to the work of the mathematical physicist John Roberts [17]. His original interest in this topic grew from his conviction that

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(strict) ω -categories constituted the appropriate algebraic structures within which to value non-abelian cohomology theories [16]. This led him to define complicial sets to be simplicial sets equipped with a distinguished set of *neutral* or *thin* simplices, called *stratified simplicial sets*, and satisfying a certain kind of *unique thin horn filler* condition. He conjectured that it should be possible to generalise the classical categorical nerve construction to provide a functor from the category of strict ω -categories to the category of complicial sets and this would provide an equivalence between these two categories.

The first step in realising his vision was made by Ross Street in his paper on orientals [20], which provided the first fully rigorous description of such a nerve construction and re-formulated Roberts' vision into a specific conjecture. More recently, the original program outlined in these papers was completed by the current author [24] who provided the first proof of the full Street–Roberts conjecture. That work demonstrates that it is indeed the case that Street's nerve construction provides the equivalence that Roberts proposed.

This result itself provides a new and powerful approach to studying strict ω -categories themselves. For instance, in [24] we show how to construct a combinatorially simple tensor product of stratified sets whose reflection into the category of strict ω -categories is the *lax Gray tensor product*. Calculations involving this latter structure are known to be complicated by the fact that it is usually presented as a colimit of strict ω -categories freely generated by geometric products of *globs*. On the other hand, if we are willing to work in the world of stratified simplicial sets then we may instead describe the corresponding tensor directly by equipping the product of underlying simplicial sets with a suitably defined set of thin simplices. In contrast this latter structure is eminently well suited to direct combinatorial calculation.

However, it was not this kind of application to strict ω -category theory which originally encouraged the current author's interest in the Street–Roberts conjecture. Instead it was piqued by parenthetical comments in Street's paper on oriented simplices [20] to the effect that we might use it as a foundation upon which to develop a useful generalisation of the theory of *bicategories* to higher dimensions. At that time no truly workable theory of such *weak ω -categories* had been constructed, although a growing group of researchers were becoming aware of the role that such structures might play in algebraic topology, theoretical physics, computer science and higher category theory itself. In brief, Street's idea was that we might obtain such a theory by again working with stratified simplicial sets but this time weakening the axioms that characterised complicial sets by only insisting that horns of the kind identified in that theory should have *some*, not necessarily unique, thin filler.

The subsequent 20 years has been a fertile one in weak ω -category theory and we might now identify in the literature three or four quite distinct approaches to defining such structures, each of which splits into a plethora of definitional sub-varieties. In this time Street's original side remark has remained largely under investigated, indeed for a number of years the current author has avoided writing up his own ideas along these lines for fear of simply launching yet another weakened higher category definition on the world. Spurred on, however, by Street's 2003 work on weak ω -categories [22], which reformulated and refined his original insight and introduced the term *weak complicial set* and Joyal's work on *quasi-categories* [11], I am now of the view that a thorough explication of this theory is well overdue.

So why might we be interested in studying weak ω -category theories based upon simplicial rather than globular geometries? From a philosophical perspective, Street himself sums the case up best in the following passage from [22]:

Simplicial sets are lovely objects about which algebraic topologists know a lot. If something is described as a simplicial set, it is ready to be absorbed into topology. Or, in other words, no matter which definition of weak ω -category eventually becomes dominant, it will be valuable to know its simplicial nerve.

In short, any weak ω -category theory worth its salt should come equipped with a simplicial nerve functor describing its place in algebraic topology. Furthermore, it is reasonable to expect that this would, at the very least, map each weak ω -category to a weak complicial set. It follows, therefore, that any study of weak complicial sets themselves will remain valuable regardless of which particular formulation of the weak ω -category notion might become dominant in the future.

More pragmatically, the answer to this question is really one of utility. As we shall see here the theory of weak complicial sets is one which immediately generalises the most widely accepted 0-trivial and 1-trivial weak ω -categorical structures (Kan sets and quasi-categories respectively) and at the same time encompasses the theory of strict ω -categories. Furthermore, we shall also demonstrate here that it supports a plethora of well behaved ω -categorical constructions, such as joins (Section 3) and Gray tensor products (Section 5), and admits a rich homotopy theory (Section 6). In a companion paper [23], we derive a nerve construction for categories enriched, in the classical sense of [13], in weak complicial sets with respect to a Gray tensor product (called *complicial Gray-categories*) which faithfully represents such structures as weak complicial sets. In particular, this demonstrates that the totality of all weak complicial sets and their homomorphisms, strong transformations, modifications and so forth is itself representable as a very richly structured (large) weak complicial set.

The actual category theory of these structures is best explored by representing weak complicial sets as certain kinds of *complicially enriched quasi-categories*. This provides us with a natural context in which to generalise traditional category theory to a kind of homotopy coherent quasi-category theory within the Quillen model category of weak complicial sets itself. This approach allows us to translate all of the basic constructions of n -category theory into the weak complicial context and at the same time to establish for it homotopical versions of the theories of discrete fibrations, Yoneda's lemma, adjunctions, limits and colimits and so forth. However, we consign this analysis to a later work.

While all of this speaks to the expressiveness of weak complicial set theory, we must also convince ourselves that it provides a strong enough framework within which to establish certain natural coherence theorems. While work in this direction is still at a relatively early stage, studies to date indicate that it is likely that a direct analogue of the well-known coherence theorems for bicategories and tricategories holds in this context as well. To be precise, there are strong reasons to suspect that every weak complicial set satisfying certain very mild conditions on its thin 1-simplices (related to our work here in Section 4) is equivalent to the nerve of some complicial Gray-category.

Herein, however, we restrict ourselves to the modest task of establishing the foundational homotopy theory of weak complicial sets upon which all of our later work in this area will be based. Section 2 introduces these structures and establishes the associated theory of anodyne extensions between stratified simplicial sets. Section 3 studies the join operation on stratified sets, introduces the corresponding décalage construction and demonstrates that these are appropriately well behaved with respect to weak compliciality. This work is then applied in Section 4 to show that we can usefully replace the condition which stipulates that weak complicial sets must have

thin fillers for outer complicial horns with one which simply states that all thin 1-simplices are actually equivalences in some suitable sense.

Section 5, which is combinatorially the most involved of this work, reintroduces the (lax) Gray tensor products of [24] and studies their properties with regard weak complicial sets and anodyne extensions. In particular, this allows us to show that we may construct weak complicial sets of homomorphisms, (lax) transformations, (lax) modifications and so forth between any pair of weak complicial sets and thereby enrich the category of these structures over itself in three distinct but related ways. Section 5.3 provides a new characterisation of strict complicial sets as those weak ones which are well-tempered, in the sense that for these thinness is a sufficient property for the detection of degenerate simplices.

Finally Section 6 draws together these threads by constructing a Quillen model structure on the category of stratified simplicial sets whose cofibrations are the inclusions and whose fibrant objects are precisely the weak complicial sets. Furthermore, we show that this is a monoidal model category with respect to the Gray tensor products studied in Section 5. Finally, we round out our presentation by localising our model structure and transporting it to the category of simplicial sets itself, in order to provide an independent construction of a model category structure on that latter category whose fibrant objects are Joyal's quasi-categories [11].

2. Introducing weak complicial sets

Here we recall the standard notation of the theory of simplicial sets, introduce their *stratified* generalisations and establish the basic machinery required to define and study *weak complicial sets*.

2.1. Stratified simplicial sets

Notation 1 (*Simplicial operators*). As usual we let Δ_+ denote the (skeletal) category of finite ordinals and order preserving maps between them and use the notation Δ to denote its full subcategory of non-zero ordinals. Following tradition we let $[n]$ denote the ordinal $n + 1$ as an object of Δ_+ and refer to arrows of Δ_+ as *simplicial operators*. We will generally use lower case Greek letters $\alpha, \beta, \gamma, \dots : [m] \rightarrow [n]$ to denote simplicial operators and let $\text{im}(\alpha)$ denote the subset $\{i \in [n] \mid \exists j \in [m]. \alpha(j) = i\} \subseteq [n]$ known as the *image* of the operator α . We will also use the following standard notation and nomenclature throughout:

- The injective maps in Δ_+ are referred to as *face operators*. For each $j \in [n]$ we use the $\delta_j^n : [n - 1] \rightarrow [n]$ to denote the *elementary* face operator distinguished by the fact that its image does not contain the integer j .
- The surjective maps in Δ_+ are referred to as *degeneracy operators*. For each $j \in [n]$ we use $\sigma_j^n : [n + 1] \rightarrow [n]$ to denote the *elementary* degeneracy operator determined by the property that two integers in its domain map to the integer j in its codomain.
- For each $i \in [n]$ the operator $\varepsilon_i^n : [0] \rightarrow [n]$ given by $\varepsilon_i^n(0) = i$ is called the *i th vertex operator* of $[n]$.
- We also use the notations $\eta^n : [n] \rightarrow [0]$ and $\iota^n : [-1] \rightarrow [n]$ to denote the unique such simplicial operators.

Unless doing so would introduce an ambiguity, we will tend to reduce notational clutter by dropping the superscripts of these elementary operators.

Notation 2 (*Simplicial sets*). The category \mathbf{Simp} of *simplicial sets* and *simplicial maps* between them is simply the functor category $[\Delta^{\text{op}}, \mathbf{Set}]$, where \mathbf{Set} denotes the (large) category of all (small) sets and functions between them. If $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ is a simplicial set then we will often simplify our notation by using X_n for the object $X([n]) \in \mathcal{M}$ and X_α for the function $X(\alpha) : X([m]) \rightarrow X([n])$. We also adopt the standard Latin notations d_i^n , s_i^n and v_i^n for the actions of the elementary simplicial operators δ_i^n , σ_i^n and ε_i^n respectively.

In practice, it is often easier to think of a simplicial set as a single set endowed with a partially defined right action of the simplicial operators. To be more precise, this description presents a simplicial set as a triple consisting of a single set X , a dimension function $\dim : X \rightarrow \mathbb{N}$, and a partial right action $x \cdot \alpha$ of $\alpha \in \text{arr}(\Delta)$ on $x \in X$ which is defined whenever the dimension of $x \in X$ equals that of the codomain of α . Under this presentation, simplicial maps become functions of underlying sets which preserve both dimension and action. We say that X is a *simplicial subset* of a simplicial set Y , denoted $X \subseteq_s Y$, if X is a subset of Y which is closed in there under the action of simplicial operators and thus inherits a simplicial set structure from it. We adopt the following traditional denotations of a few fundamental simplicial sets:

- The *standard n -simplex* $\Delta[n]$ which is the representable simplicial set on $[n]$, whose r -simplices are operators $\alpha : [r] \rightarrow [n] \in \Delta$ acted upon by right composition.
- The *boundary of the n -simplex* $\partial\Delta[n]$ which is the simplicial subset of $\Delta[n]$ of those simplices $\alpha : [r] \rightarrow [n]$ which are **not** surjective. Notice that the boundary of the 0-simplex $\partial\Delta[0]$ is simply the empty stratified set \emptyset .
- The *$(n-1)$ -dimensional k -horn* $\Lambda^k[n]$ which is the simplicial subset of $\Delta[n]$ consisting of those simplices $\alpha : [r] \rightarrow [n]$ for which there is some $i \in [n]$ which is neither in the image of α nor equal to k (that is for which $[n] \neq \text{im}(\alpha) \cup \{k\}$). In other words, this is the smallest simplicial subset of $\Delta[n]$ containing the set of $(n-1)$ -faces $\{\sigma_i^n : i \in [n] \setminus \{k\}\}$.

We say that a simplex x of a simplicial set X is *degenerate* iff there is some non-identity degeneracy operator α and a simplex $x' \in X$ such that $x = x' \cdot \alpha$. More specifically we say that x is *degenerate at k* if $x = x' \cdot \sigma_k$ for some simplex $x' \in X$, in which case we would have $x' = x \cdot \delta_k = x \cdot \delta_{k+1}$. The *Eilenberg–Zilber lemma* tells us that every simplex $y \in X$ may be represented uniquely as $y = x \cdot \beta$ where β is a degeneracy operator and x is a **non-degenerate** simplex.

Finally, recall that *Yoneda’s lemma* for simplicial sets tells us that there exists a natural bijection between the n -simplices of a simplicial set X and simplicial maps $\Delta[n] \rightarrow X$. This identifies $x \in X_n$ with the simplicial map $\ulcorner x \urcorner$ that carries the simplex $\alpha \in \Delta[n]$, which is simply a simplicial operator with codomain $[n]$, to the simplex $\ulcorner x \urcorner(\alpha) \stackrel{\text{def}}{=} x \cdot \alpha$ in X .

Notation 3. We introduce the following notations to denote the simplices of the standard simplex $\Delta[1]$:

- $0^r : [r] \rightarrow [1]$ is the operator which maps each $i \in [r]$ to $0 \in [1]$.
- $1^r : [r] \rightarrow [1]$ is the operator which maps each $i \in [r]$ to $1 \in [1]$.
- $\rho_i^r : [r] \rightarrow [1]$ ($1 \leq i \leq r$) is the operator defined by

$$\rho_i^r(j) = \begin{cases} 0 & \text{if } j < i, \\ 1 & \text{if } j \geq i. \end{cases}$$

As above, we shall adopt the convention of omitting the superscripts on these operators unless doing so would introduce an ambiguity. Later on it will become convenient to index the r -simplices of $\Delta[1]$ using the *doubly pointed* set $\llbracket r \rrbracket \stackrel{\text{def}}{=} \{-, +, 1, 2, \dots, r\}$, by letting $\rho_-^r = 0^r$, $\rho_+^r = 1^r$ and defining ρ_i^r as above for an arbitrary integer (non-point) in $\llbracket r \rrbracket$.

Observation 4 (*Nerves of categories*). We shall also assume that the reader is familiar with the classical *nerve construction* which functorially associates a simplicial set $N(\mathbb{C})$ to each category \mathbb{C} . This is formed by regarding the ordered sets $[n]$ to be categories in the usual way and applying *Kan's construction* [12] to the inclusion of Δ as a full subcategory into $\underline{\text{Cat}}$ (the category of small categories), to obtain an adjoint pair:

$$\begin{array}{ccc} & \text{F} & \\ \text{Cat} & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & \text{Simp.} \\ & \text{N} & \end{array}$$

In other words, the n -simplices of $N(\mathbb{C})$ are functors $f: [n] \rightarrow \mathbb{C}$ upon which simplicial operators act by pre-composition.

Definition 5 (*Stratified simplicial sets*). A *stratification* on a simplicial set X is a subset² tX of its simplices satisfying the conditions that

- no 0-simplex of X is in tX , and
- all of the degenerate simplices of X are in tX .

A *stratified set* is a pair (X, tX) consisting of a simplicial set X and a chosen stratification tX the elements of which we call *thin* simplices. In practice, we will elect to notionally confuse stratified sets with their underlying simplicial sets X, Y, Z, \dots and uniformly adopt the notation tX, tY, tZ, \dots for corresponding sets of thin simplices. *Then, where disambiguation is required, we use the notation $\tilde{X}, \tilde{Y}, \tilde{Z}, \dots$ to denote the underlying simplicial sets of these stratified sets.*

A *stratified map* $f: X \rightarrow Y$ is simply a simplicial map of underlying simplicial sets which *preserves thinness* in the sense that for all $x \in tX$ we have $f(x) \in tY$. Identities and composites of stratified maps are clearly stratified maps, from which it follows that we have a category $\underline{\text{Strat}}$ of stratified sets and maps.

Observation 6 (*Stratifying simplicial sets*). The functor from $\underline{\text{Strat}}$ to $\underline{\text{Simp}}$ which forgets stratifications has both a left and a right adjoint, which assign to a simplicial set its minimal and maximal stratification respectively. We will implicitly promote any simplicial set $X \in \underline{\text{Simp}}$ to a stratified set using the (left adjoint) minimal stratification, under which its sets of degenerate and thin simplices coincide, and thereby regard $\underline{\text{Simp}}$ as a full subcategory of $\underline{\text{Strat}}$. In particular, the representable simplicial sets $\Delta[n] \in \underline{\text{Simp}}$ provide us with geometrical models for the standard simplices in $\underline{\text{Strat}}$.

² Note that tX is merely a subset of X , **not** a simplicial subset, in general it will not be closed in X under the action of simplicial operators.

Definition 7 (*Stratified subsets, inverse and direct images*). Suppose that U and X are stratified sets, then we say that U is a *stratified subset* of X , denoted $U \subseteq_s X$, if \tilde{U} is a simplicial subset of \tilde{X} and its stratification tU is a subset of tX . If $f : X \rightarrow Y$ is a stratified map then the:

- *direct image* of the stratified subset $U \subseteq_s X$ along f is the stratified subset $f(U) \subseteq_s Y$ with underlying simplicial set $\{f(x) \mid x \in U\}$ and in which $y \in f(U)$ is thin iff there is some $x \in tU$ with $f(x) = y$.
- *inverse image* of the stratified subset $V \subseteq_s Y$ along f is the stratified subset $f^{-1}(V) \subseteq_s X$ with underlying simplicial set $\{x \in X \mid f(x) \in V\}$ and in which $x \in f^{-1}(V)$ is thin iff $f(x)$ is thin in V .

Definition 8 (*Inclusions of stratified sets*). We call the monomorphisms in Strat *stratified inclusions*, and, by custom, we will generally draw these using arrows with hooked domains $i : X \hookrightarrow Y$. It is clear that these are precisely those stratified simplicial maps which act injectively on simplices. A stratified subset X of Y clearly gives rise to a corresponding stratified inclusion which we denote by $X \hookrightarrow_s Y$. Indeed, wherever necessary we may always replace an arbitrary stratified inclusion by an isomorphic stratified subset inclusion.

Definition 9 (*Stratified subsets, regularity and entirety*). We say that a stratified subset X of Y is

- *regular*, denoted $X \subseteq_r Y$, if $tX = \tilde{X} \cap tY$, and
- *entire*, denoted $X \subseteq_e Y$, if $\tilde{X} = \tilde{Y}$,

wherein \tilde{X} and \tilde{Y} denote the underlying simplicial sets of X and Y respectively (as discussed in Definition 5).

The terms *regular subset* and *entire subset* will always be taken to denote stratified subsets which possess the appropriate property. If W is a subset of simplices of the stratified set X then the stratified (resp. regular or entire) subset of X *generated* by W is defined to be the smallest such stratified subset of X which contains W .

Extending these definitions to all stratified maps, we say that $f : X \rightarrow Y \in \text{Strat}$ is *regular* if it *reflects thin simplices*, meaning that whenever $f(x)$ is thin in Y it follows that x is thin in X , and *entire* if it is surjective on simplices.

A stratified map $f : X \rightarrow Y$ admits two well-behaved canonical factorisations:

- *regular image factorisation* $X \xrightarrow{f_e} \text{im}_r(f) \hookrightarrow_s Y$ in which the stratified map f_e is entire and $\text{im}_r(f)$, the *regular image* of f , is the regular subset of Y whose set of simplices is $\{y \in Y \mid \exists x \in X. f(x) = y\}$;
- *entire coimage factorisation* $X \hookrightarrow_s \text{coim}_e(f) \xrightarrow{f_r} Y$ in which the stratified map f_r is regular and $\text{coim}_e(f)$, the *entire coimage* of f , is the entire superset of X whose thin simplices are those $x \in X$ for which $f(x)$ is thin in Y .

Notation 10 (*Complicial simplices and horns*). We introduce a few other stratified sets which will take on particular importance in our deliberations later on:

- The *standard thin n -simplex* $\Delta[n]_t$ constructed from $\Delta[n]$ by making thin its unique non-degenerate n -simplex $\text{id}_{[n]} : [n] \rightarrow [n] \in \Delta[n]$.

- The k -complicial n -simplex $\Delta^k[n]$ constructed from $\Delta[n]$ by making thin all those simplices $\alpha: [r] \rightarrow [n]$ whose image contains the set of integers $\{k-1, k, k+1\} \cap [n]$. Non-degenerate simplices satisfying this latter condition are said to be k -admissible.
- The $(n-1)$ -dimensional k -complicial horn $\Lambda^k[n]$ which is the regular subset of $\Delta^k[n]$ of those simplices $\alpha: [r] \rightarrow [n]$ for which the set $[n] \setminus (\text{im}(\alpha) \cup \{k\})$ is non-empty. In other words, this is the regular subset of $\Delta^k[n]$ generated by its set $\{\delta_i \mid i \in [n] \setminus \{k\}\}$ of all $(n-1)$ -faces except δ_k .
- The stratified set $\Delta^k[n]''$ and its regular subset $\Lambda^k[n]'$ which are obtained from $\Delta^k[n]$ and $\Lambda^k[n]$ (respectively) by making all $(n-1)$ -simplices thin.
- The union $\Delta^k[n]' \stackrel{\text{def}}{=} \Delta^k[n] \cup \Lambda^k[n]' \subseteq_e \Delta^k[n]''$ which may be constructed from $\Delta^k[n]$ by making thin the $(n-1)$ -simplices δ_{k-1}^n and δ_{k+1}^n .

While the stratifications of these complicial simplices may seem a little less than intuitive, they are however fundamental to much of the theory that follows. Motivation for these choices is provided by the various works of Roberts [16] and [17], Street [20] and [21] and Verity [24].

Observation 11 (*k-admissibility recast*). It is sometime useful to recast the definition of k -admissibility slightly. To that end, it is easily shown that a non-degenerate r -simplex $\alpha \in \Delta[n]$ is k -admissible if and only if there exists some $l \in [r]$ such that $\alpha(i) = k + i - l$ for each $i \in [r] \cap \{l-1, l, l+1\}$.

Observation 12 (*Strat as a LFP quasi-topos*). The full subcategory $t\Delta$ of standard simplices and standard thin simplices is dense in Strat (cf. Chapter 5 of Kelly [13]), thereby providing us with a reflective full embedding of Strat into the presheaf category $[t\Delta^{\text{op}}, \text{Set}]$. More explicitly, $t\Delta$ may be obtained from Δ by appending extra objects $[n]_l$ for $n = 1, 2, \dots$ and extra operators $\zeta_k^n: [n+1]_l \rightarrow [n]$ and $\varphi^n: [n] \rightarrow [n]_l$ satisfying the relations $\zeta_k^n \circ \varphi^{n+1} = \sigma_k^n$. A presheaf $F \in [t\Delta^{\text{op}}, \text{Set}]$ is isomorphic to some stratified set if and only if it maps each operator $\varphi^n: [n] \rightarrow [n]_l$ to a monomorphism in Set. It follows that the category Strat is locally finitely presentable, since it is equivalent to the category of models for a finite limit sketch on $t\Delta$.

The utility of this observation is immediately clear, for instance it tells us that Strat has limits which are calculated pointwise, colimits which are constructed in $[t\Delta^{\text{op}}, \text{Set}]$ and then reflected into Strat and that its finitely presented objects are those stratified sets with only a finite number of non-degenerate simplices. Furthermore, as observed by Street in [22], the left adjoint to the inclusion Strat $\hookrightarrow [t\Delta^{\text{op}}, \text{Set}]$ preserves pullbacks of pairs of morphisms into (images of) stratified sets from which it follows that Strat is a quasi-topos. In other words, for each stratified set X the slice category Strat/ X is cartesian closed and Strat has a classifier for regular subobjects.

Notation 13 (*Skeleta and superstructures*). We say that a stratified set is n -skeletal if all of its simplices of dimension greater than $n \in \mathbb{N}$ are degenerate. The n -skeleton $\text{sk}_n(X)$ of a stratified set X is its regular subset consisting of those of its simplices whose faces of dimension greater than n are all degenerate. This construction provides us with an endo-functor of Strat whose range is the full subcategory of n -skeletal stratified sets and which has a right adjoint ck_n called the n -coskeleton functor.

Playing the same game with thinness, we say that a stratified set is n -trivial if all of its simplices of dimension greater than n are thin. The n -trivialisation $\text{th}_n(X)$ of a stratified set X is its

entire superset constructed by making thin all of its simplices of dimension greater than n . Again this construction provides us with an endo-functor of Strat whose range is the full-subcategory of n -trivial stratified sets and which has a right adjoint sp_n called the n -superstructure functor. The n -superstructure $\mathrm{sp}_n(X)$ may be realised as the regular subset of X of those simplices whose faces of dimension greater than n are all thin.

2.2. Weak complicial sets

Now we are ready to embark on defining and studying weak complicial sets:

Notation 14 (*Lifting problems and properties*). A commutative square in some category \mathcal{C}

$$\begin{array}{ccc} U & \xrightarrow{u} & E \\ i \downarrow & \nearrow & \downarrow p \\ V & \xrightarrow{v} & A \end{array}$$

is called a *lifting problem* from i to p and it is said to *have a solution* if there exists some diagonal map (dotted in the diagram) which makes both triangles commute. When such a solution exists we say that i has the *left lifting property* (LLP) with respect to p or that p had the *right lifting property* (RLP) with respect to i .

We say that an object $C \in \mathcal{C}$ has the RLP with respect to the morphism $i : U \rightarrow V$ iff the unique map $C \rightarrow 1$ into the terminal object of \mathcal{C} enjoys that property. In such a case, a lifting problem amounts to a morphism $u : U \rightarrow C$ and a solution to this is simply a morphism $\bar{u} : V \rightarrow C$ for which $\bar{u} \circ i = u$.

Definition 15 (*Elementary anodyne extensions and weak complicial sets*). The set of *elementary anodyne extensions* in Strat consists of two families of subset inclusions:

- $\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ for $n = 1, 2, \dots$ and $k \in [n]$, these are called *complicial horn extensions*, and
- $\Delta^k[n]' \xrightarrow{\subseteq_e} \Delta^k[n]''$ for $n = 2, 3, \dots$ and $k \in [n]$, these are called *complicial thinness extensions*.

We classify these elementary anodyne extensions into two sub-classes, the *inner* ones for which the index k satisfies $0 < k < n$ and the remaining *left* and *right outer* ones for which $k = 0$ or $k = n$ respectively. Now we say that a stratified set A is a

- *weak inner complicial set* if it has the RLP with respect to all inner elementary anodyne extensions;
- *weak left (resp. right) complicial set* if it is a weak inner complicial set which also has the RLP with respect to all left (resp. right) outer elementary anodyne extensions;
- *weak complicial set* if it has the RLP with respect to all elementary anodyne extensions.

Informally we might simply say that a weak complicial set has *fillers* for all complicial horns.

Example 16 (*Kan complexes and Joyal's quasi-categories*). The theory of weak complicial sets generalises and subsumes those of Kan complexes and Joyal's quasi-categories. In particular, if X is a simplicial set then it is:

- a Kan complex iff $\mathrm{th}_0(X)$ is a weak complicial set, and
- a quasi-category iff it admits some 1-trivial stratification which makes it into a weak complicial set.

The first of these observations is trivial, the second is a direct consequence of Theorem 1.3 in Joyal's paper on quasi-categories [11]. We return to this example in Section 4, where we generalise and reprove Joyal's result in the current context.

Example 17 (*Complicial sets*). The notion of *complicial set*, which is the object of study of [24], corresponds to a strict form of the structures defined above. Weak complicial sets satisfy a horn filler condition that requires the existence of a thin filler for each complicial horn. Complicial sets, on the other hand, satisfy a stronger 'unique horn filler' condition under which we ask for such fillers to be uniquely determined.

While the literal definition of these stronger structures, as given in [24], does not actually require outer complicial horns to have fillers, its Lemma 163 demonstrates that any left (resp. right) outer complicial n -simplex x in a complicial set is degenerate at 0 (resp. $n - 1$), that is to say $x = x \cdot \sigma_0$ (resp. $x = x \cdot \sigma_{n-1}$) as discussed in Definition 2. From this fact it is easily demonstrated that in a complicial set any outer complicial horn may be (uniquely) filled by a degenerate simplex. It follows that any complicial set is actually a weak complicial set. A converse to this result, providing an alternative characterisation of complicial sets amongst the weak complicial sets, may be found in Theorem 80.

We will sometimes say that the complicial sets of [24] are *strict* in order to differentiate them more clearly from the far more general weak complicial sets of this paper.

Example 18 (*Stratifying ω -categorical nerves*). The combinatorial calculations of Street [21] demonstrate that the nerve $N_\omega(\mathbb{C})$ of any (strict) ω -category \mathbb{C} may be made into a (strict) complicial set by endowing it with the *Roberts stratification* in which the commutative simplices are thin. However, the same calculations may be pushed a little further to show that $N_\omega(\mathbb{C})$ is also made into a (generally non-strict) weak complicial set, denoted $N_\omega^e(\mathbb{C})$, by endowing it with the stratification under which a n -simplex is thin if it maps the unique non-trivial n -cell of the n th oriental \mathcal{O}_n to an ω -categorical n -equivalence in \mathbb{C} . The precise formulation and proof of this fact, which we shall not require further here, is a matter of direct ω -categorical calculation which we leave, as an exercise, to the reader.

Example 19 (*Nerves of complicial Gray-categories*). As discussed later, in Section 5, the cartesian product of stratified sets plays the role of the *Gray tensor product* in the theory of weak complicial sets. Consequently, it is natural to define a *complicial Gray-category* to be a category enriched over the cartesian category of weak complicial sets. In the companion paper [23] we generalise the *homotopy coherent nerve* construction of Cordier and Porter [3] to provide a nerve functor which faithfully represents such complicial Gray-categories as weak complicial sets. Later in this work we show that the category of weak complicial sets is itself a complicial Gray-category to which we may apply this nerve construction and thereby represent the universe of all (small) weak complicial sets canonically as a (large) weak complicial set.

Notation 20 (*Fibrations and cofibrations*). If I is a set of morphisms in some category \mathcal{C} then we adopt the following standard notations:

- $\text{cell}(I)$ denotes the *cellular completion* of I , that is the closure of the class of pushouts of elements of I under transfinite composition, whose elements are called *relative I -cell complexes*,
- $\text{cof}(I)$ denotes the closure of $\text{cell}(I)$ under retraction, whose elements are called *I -cofibrations*, and
- $\text{fib}(I)$ denotes the class of maps which have the RLP with respect to I , whose elements are called *I -fibrations*. We say that an object A is *I -fibrant* if the unique morphism $A \rightarrow 1$ to the terminal object is an I -fibration.

These definitions ensure that each I -fibration $p: A \rightarrow B$ has the RLP with respect to any I -cofibration $i: U \rightarrow V$.

We will assume from hereon that the reader is familiar the basic properties of classes of fibrations and cofibrations in a form that usually accompanies modern presentations of categorical homotopy theory. If this is not the case then any one of the commonly cited introductions to the basic theory of Quillen model categories should provide the suitable background. Certainly a familiarity with Dwyer and Spaliński's excellent survey article [7] would suffice for our purposes here.

Example 21 (*Cellular completion and stratified inclusions*). Our construction of the colimits of Strat, as given in Observation 57, immediately implies that the colimits of Strat are preserved by the stratification forgetting functor $(-)^{\sim}: \text{Strat} \rightarrow \text{Simp}$. Furthermore we know, from Definition 8 that a stratified map is a stratified inclusion if and only if its underlying action on simplicial sets is an inclusion.

Of course inclusions and colimits in Simp are determined pointwise in Set, from which it follows that the class of inclusions is closed in Simp under pushout, transfinite composition and retraction. It follows, therefore, from the last paragraph, that the class of stratified inclusions is closed in Strat under these same constructions.

Furthermore, we may extend a classical simplicial result to show that any stratified inclusion $X \xrightarrow{\subseteq_s} Y$ can be expressed as the countable composite of the sequence of skeleta inclusions (cf. Notation 13, p. 1088)

$$X \xrightarrow{\subseteq_s} X \cup \text{sk}_0(Y) \xrightarrow{\subseteq_s} X \cup \text{sk}_1(Y) \xrightarrow{\subseteq_s} \cdots \cdots X \cup \text{sk}_n(Y) \xrightarrow{\subseteq_s} \cdots \cdots$$

and that each inclusion $X \cup \text{sk}_{n-1}(Y) \xrightarrow{\subseteq_s} X \cup \text{sk}_n(Y)$ in this sequence can be constructed as a pushout of copies of the boundary and thin simplex inclusions $\partial \Delta[n] \hookrightarrow \Delta[n]$ and $\Delta[n] \hookrightarrow \Delta[n]_t$. It follows therefore that the class of all stratified inclusions is the cellular completion of the set of boundary and thin simplex inclusions:

$$\{\partial \Delta[n] \xrightarrow{\subseteq_r} \Delta[n] \mid n = 0, 1, \dots\} \cup \{\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_t \mid n = 1, 2, \dots\}.$$

Definition 22 (*Anodyne extensions and complicial fibrations*). We say that a stratified inclusion $e: U \hookrightarrow V \in \text{Strat}$ is an (*inner*) *anodyne extension* if it is in the cellular completion of the set of

elementary (inner) anodyne extensions. Correspondingly, we say that a stratified map $p : E \rightarrow A$ is a (inner) *complicial fibration* if it is a fibration with respect to the set of elementary (inner) anodyne extensions.

We also sometimes say that e is an *right* (resp. *left*) anodyne extension if it is in the cellular completion of the union of the sets of inner and right outer (resp. left outer) anodyne extensions. Members of the corresponding class of fibrations are known as *right* (resp. *left*) *complicial fibrations*.

Of course, we may rephrase Definition 15 in these terms by saying that A is a weak (inner, left, right) complicial set iff the unique map $p : A \rightarrow 1$ into the terminal stratified set is an (inner, left, right) complicial fibration.

Definition 23 (*Thinness extensions*). We say that a stratified map $U \xrightarrow{e} V$ is a *thinness extension* if it is both an anodyne extension and an entire inclusion. By definition all elementary thinness extensions and any (transfinite) composite of pushouts of such things are also thinness extensions.

In general it is clearly that solutions of lifting problems whose domains are entire maps are unique. Furthermore, this uniqueness property immediately implies that if a stratified map A has the RLP with respect to some entire map then any stratified map $p : A \rightarrow B$ also has that property. Consequently, it follows that any stratified map whose domain is a weak complicial set has the RLP with respect to any thinness extension.

Recall 24 (*Glueing squares*). A *glueing square* is a commutative square in some category which is both a pushout and a pullback. When constructing anodyne extensions we will often construct the pushouts we need as glueing squares, using the simple observation that if $i : U \hookrightarrow X$ is a stratified inclusion and $V \subseteq_s X$ is a stratified subset of its codomain then the first of the following squares

$$\begin{array}{ccc} f^{-1}(V) \hookrightarrow V \\ \subseteq_s \downarrow \lrcorner \quad \downarrow \subseteq_s \\ X \xrightarrow{i} f(X) \cup V \end{array} \qquad \begin{array}{ccc} U \cap V \xrightarrow{\subseteq_s} V \\ \subseteq_s \downarrow \lrcorner \quad \downarrow \subseteq_s \\ U \xrightarrow{\subseteq_s} U \cup V \end{array}$$

of inclusions is a glueing square in Strat. When the inclusion i is actually a subset inclusion $U \xrightarrow{\subseteq_s} X$, this may be re-drawn to give the glueing square to its right.

For instance, to prove that the regular inclusion $\Lambda^k[n]' \xrightarrow{\subseteq_r} \Delta^k[n]''$ is an anodyne extension we start with the diagram:

$$\begin{array}{ccccc} \Lambda^k[n] & \xrightarrow{\subseteq_r} & \Delta^k[n] & & \\ \subseteq_e \downarrow & & \downarrow \subseteq_e & & \\ \Lambda^k[n]' & \xrightarrow{\subseteq_r} & \Delta^k[n]' & \xrightarrow{\subseteq_e} & \Delta^k[n]'' \end{array}$$

Applying the observation of the last paragraph, we show that the square here is a glueing square in Strat since $\Lambda^k[n] = \Lambda^k[n]' \cap \Delta^k[n]$ and $\Delta^k[n]' = \Lambda^k[n]' \cup \Delta^k[n]$. Its upper horizontal is a

complicial horn extension so it follows that its lower horizontal is an anodyne extension, which we compose with the elementary thinness extension to its right to obtain the desired presentation of $\Delta^k[n]' \xrightarrow{\subseteq_r} \Delta^k[n]''$ as an anodyne extension. We call this inclusion an (*inner*) *thin horn extension*.

Lemma 25 (*Superstructures of weak complicial sets*). For each $n \in \mathbb{N}$ the n -trivialisation functor th_n (cf. Notation 13, p. 1088) preserves (*inner*) anodyne extensions. It follows that its right adjoint, the superstructure functor sp_n , preserves (*inner*) complicial fibrations and (*inner*) weak complicial sets.

Proof (essentially Lemmas 150 and 171 of [24]). Since th_n is a left adjoint it preserves all colimits and so it is enough to check that it maps each *elementary* (*inner*) anodyne extension to an (*inner*) anodyne extension. Considering cases:

$n \geq m - 1$. Observe that each of the stratified sets $\Delta^k[m]$, $\Delta^k[m]'$ and $\Delta^k[m]''$ is $(m - 1)$ -trivial, so if $n \geq m - 1$ then they are also n -trivial. It follows that the endo-functor th_n maps each of these sets, and thus each of the elementary anodyne extensions $\Delta^k[m] \xrightarrow{\subseteq_r} \Delta^k[m]$ and $\Delta^k[m]' \xrightarrow{\subseteq_e} \Delta^k[m]''$, to itself.

$n < m - 1$. Then we know that $\Delta^k[m]$, $\Delta^k[m]'$ and $\Delta^k[m]''$ only differ in as much as they have different sets of thin simplices at dimension $m - 1$ and consequently, since $n < m - 1$, we know that $\text{th}_n(\Delta^k[m]) = \text{th}_n(\Delta^k[m]') = \text{th}_n(\Delta^k[m]'')$. It follows that the functor th_n maps the elementary thinness extension $\Delta^k[m]' \xrightarrow{\subseteq_e} \Delta^k[m]''$ to the identity on $\text{th}_n(\Delta^k[m])$.

Finally, observe that when $n < m - 1$ we know that $\Delta^k[m]''$ is an entire subset of $\text{th}_n(\Delta^k[m])$ and that its union with the regular subset $\text{th}_n(\Delta^k[m]) \subseteq_r \text{th}_n(\Delta^k[m])$ is equal to $\text{th}_n(\Delta^k[m])$ itself. What is more, the intersection of these subsets $\Delta^k[m]'' \cap \text{th}_n(\Delta^k[m])$ is equal to the regular subset $\Delta^k[m]' \subseteq_r \Delta^k[m]''$ and it follows that

$$\begin{array}{ccc} \Delta^k[m]' & \xrightarrow{\subseteq_r} & \Delta^k[m]'' \\ \subseteq_s \downarrow & & \downarrow \subseteq_s \\ \text{th}_n(\Delta^k[m]) & \xrightarrow[\subseteq_r]{} & \text{th}_n(\Delta^k[m]) \end{array}$$

is a glueing square Strat. We demonstrated that the upper horizontal map in this square is an anodyne extension in Recollection 24, so it follows that its pushout the lower horizontal is also an anodyne extension as required.

The second sentence of the statement follows directly from the first under the adjunction $\text{th}_n \dashv \text{sp}_n$.

Observation 26 (*Alternating duals of weak complicial sets*). The canonical idempotent endo-functor $(-)^{\circ} : \Delta_+ \rightarrow \Delta_+$ which acts as the identity on objects and maps an operator $\alpha : [n] \rightarrow [m]$ to the operator defined by $\alpha^{\circ}(i) = m - \alpha(n - i)$ may be extended to a idempotent endo-functor on the category of stratified sets Strat called the *alternating dual*. This carries a

stratified set X to X° which has the same sets of simplices and thin simplices as X but has a dual action $*$ under which a simplicial operator α acts on a simplex x according to the formula $x * \alpha = x \cdot \alpha^\circ$.

The action of $(-)^{\circ}$ on operators provides us with a canonical isomorphism between the standard n -simplex $\Delta[n]$ and its dual $\Delta[n]^{\circ}$ and it is clear that this underlies an isomorphism between the complicial simplex $\Delta^{n-k}[n]$ and the dual $\Delta^k[n]^{\circ}$. Consequently, we see that on taking duals of the elementary anodyne extensions $\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ and $\Delta^k[n]' \xrightarrow{\subseteq_e} \Delta^k[n]''$ we obtain inclusions which are isomorphic to $\Delta^{n-k}[n] \xrightarrow{\subseteq_r} \Delta^{n-k}[n]$ and $\Delta^{n-k}[n]' \xrightarrow{\subseteq_e} \Delta^{n-k}[n]''$ respectively.

As an idempotent functor the alternating dual is its own (left and right) adjoint, so in particular it preserves the colimits of Strat and it follows that we may extend the result of the last paragraph to demonstrate the preservation of all (inner) anodyne extensions. Furthermore, applying this adjointness property directly to the right lifting properties that define weak (inner)complicial sets and (inner) complicial fibrations, we see that these are also preserved by taking alternating duals.

2.3. Generalised horns

Before moving on we prove a simple technical lemma, of some use later on, which shows that the inclusions associated with certain kinds of generalised horns are (inner) anodyne extensions. It should be noted that this is not the most general result of this kind possible and more powerful results of this nature may be found in Verity [24] or Ehlers and Porter [9]. However, our simpler result below is exactly what we will require in the sequel.

Definition 27. Suppose that $\vec{k} = \{k_1, k_2, \dots, k_t\} \subset [n]$ is a non-empty family of integers with $k_i + 1 < k_{i+1}$ for each $i = 1, 2, \dots, t - 1$, then we say that an entire superset N of the standard simplex $\Delta[n]$ is a \vec{k} -complicial n -simplex if it satisfies the following conditions for each $k_i \in \vec{k}$ and each k_i -admissible simplex $\alpha \in \Delta[n]$ (cf. Notation 10, p. 1087):

- (a) α is thin in N , and
- (b) if $l \in [r]$ is the (unique) integer such that $\alpha(l) = k_i$ and $\alpha \circ \delta_l$ is thin in N then so is $\alpha \circ \delta_j$ for each $j \in \{l - 1, l + 1\} \cap [r]$.

Notice that if N and N' are two \vec{k} -complicial n -simplices then their intersection $N \cap N'$ is also a \vec{k} -complicial n -simplex. It follows that there is a minimal stratification which makes $\Delta[n]$ into such a \vec{k} -complicial n -simplex which we call $\Delta^{\vec{k}}[n]$.

The $(n - 1)$ -dimensional \vec{k} -complicial horn $\Lambda^{\vec{k}}N$ is simply the regular subset of N of those simplices $\alpha : [r] \rightarrow [n]$ for which there is some $i \in [n]$ which is neither in the image of α nor in the set \vec{k} (that is for which $[n] \neq \text{im}(\alpha) \cup \vec{k}$). In other words, this is the regular subset of N generated by the set of $(n - 1)$ -simplices $\{\delta_i \in \Delta[n] \mid i \in [n] \setminus \vec{k}\}$. We say that $\Lambda^{\vec{k}}N$ is an *inner generalised horn* if $0 < k_1$ and $k_t < n$.

Lemma 28 (Generalised horn lemma). If N is a (inner) \vec{k} -complicial n -simplex then the associated horn inclusion $\Lambda^{\vec{k}}N \xrightarrow{\subseteq_r} N$ is an (inner) anodyne extension.

Proof. Our proof is by induction on the length of \vec{k} . For the base case, if $\vec{k} = \{k\}$ then our generalised horns are no more than ordinary k -complicial horns with extra thin simplices. To be precise, there are two possibilities for the $(n-1)$ -simplex $\delta_k \in \Delta[n]$:

Case (i). It is *not* thin in N , in which case $\Delta^k[n] \subseteq_e N$ (by condition (a) of Definition 27), $\Delta^k[n] \cap \Lambda^{\vec{k}}N = \Lambda^k[n]$ and $\Delta^k[n] \cup \Lambda^{\vec{k}}N = N$ (since δ_k is not thin in N) so we get a glueing square which displays $\Lambda^{\vec{k}}N \xrightarrow{\subseteq_r} N$ as a pushout of the complicial horn $\Lambda^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$.

Case (ii). It is thin in N , in which case condition (b) of Definition 27 applied to the n -simplex $\text{id}_{[n]}: [n] \rightarrow [n]$ ensures that we actually have $\Delta^k[n]'' \subseteq_e N$ and consequently that $\Delta^k[n]'' \cap \Lambda^{\vec{k}}N = \Lambda^k[n]'$ and $\Delta^k[n]'' \cup \Lambda^{\vec{k}}N = N$ so we get a glueing square which displays $\Lambda^{\vec{k}}N \xrightarrow{\subseteq_r} N$ as a pushout of the (inner) thin horn extension $\Lambda^k[n]' \xrightarrow{\subseteq_r} \Delta^k[n]''$ of Observation 24.

In either case it follows that the inclusion of the statement is an (inner) anodyne extension (as a pushout of such).

To establish the inductive case, suppose that the result holds for the vector $\vec{k} = \{k_1, k_2, \dots, k_t\}$ and consider the extended vector $\vec{k}' = \vec{k} \cup \{k\}$ where $k_t + 1 < k \in [n]$. Suppose also that N satisfies the conditions of Definition 27 with respect to \vec{k}' . The $(n-1)$ -simplex $\delta_k \in N$ corresponds to a stratified inclusion $\lceil \delta_k \rceil: \Delta[n-1] \hookrightarrow N$ (by Yoneda's lemma) which we may factor, as in Definition 9, to obtain an entire superset M of $\Delta[n-1]$ and a regular inclusion $M \hookrightarrow N$. Explicitly, the simplex $\alpha \in \Delta[n-1]$ is thin in M iff $\delta_k \circ \alpha$ is thin in N , from which description it is a matter of routine verification, using the fact that $k_t + 1 < k$, to check that M also satisfies the conditions of Definition 27 with respect to \vec{k} . By construction, the union of the image of $M \hookrightarrow N$ and the horn $\Lambda^{\vec{k}}N \subseteq_r N$ is the more complete generalised horn $\Lambda^{\vec{k}}N \subseteq_r N$ and, furthermore, the subset $\Lambda^{\vec{k}}M \subseteq_r M$ is easily seen to be the inverse image of the regular subset $\Lambda^{\vec{k}}N \subseteq_r N$ along that inclusion. So we obtain a commutative diagram

$$\begin{array}{ccccc} \Lambda^{\vec{k}}M & \xrightarrow{\subseteq_r} & M & & \\ \downarrow & & \downarrow & & \\ \Lambda^{\vec{k}'}N & \xrightarrow{\subseteq_r} & \Lambda^{\vec{k}}N & \xrightarrow{\subseteq_r} & N \end{array}$$

in which the left-hand square is a glueing square and the upper horizontal and right-hand lower horizontal maps are both (inner) anodyne extensions by the induction hypothesis. It follows that the lower left horizontal is also an (inner) anodyne extension, since it is a pushout of such, and thus that its composite $\Lambda^{\vec{k}'}N \xrightarrow{\subseteq_r} N$ with the inclusion to its right is also an (inner) anodyne extension as required. \square

Corollary 29. *If N is an (inner) \vec{k} -complicial n -simplex then the entire inclusion $\Lambda^{\vec{k}}N \cup \Delta^{\vec{k}}[n]N \xrightarrow{\subseteq_e} N$ is an (inner) anodyne extension.*

Proof. A routine reprise of the method used in the proof of the last lemma, replacing pushouts of horn extensions by pushouts of related thinness extensions wherever necessary. We leave the details to the reader. \square

3. Joins of stratified sets

Here we generalise the ever useful simplicial join operation (see for instance [8] or [11]) to stratified sets and prove that it gives rise to décalage constructions which are well behaved with respect to weak complicity.

3.1. Augmented simplicial sets and the join construction

Recall 30 (*Ordinal sum*). The ordinal sum functor $\oplus: \Delta_+ \times \Delta_+ \rightarrow \Delta_+$ is defined on objects by letting $[n] \oplus [m] = [n + m + 1]$ and defining the sum of two operators $\alpha: [n] \rightarrow [n']$ and $\beta: [m] \rightarrow [m']$ by

$$\alpha \oplus \beta(k) = \begin{cases} \alpha(k) & \text{if } k \leq n, \\ \beta(k - n - 1) + n' + 1 & \text{otherwise.} \end{cases}$$

This functor makes Δ_+ into a strict monoidal category, whose unit is the empty ordinal $[-1]$.

Simplicial joins are constructed by extending \oplus to the category of *augmented* simplicial sets $\text{Simp}_+ = [\Delta_+^{\text{op}}, \text{Set}]$. Correspondingly, stratified joins are defined most naturally on augmented stratified sets.

Definition 31 (*Augmented stratified sets*). An *augmented stratified set* X consists of an augmented simplicial set equipped with a stratification $tX \subseteq X$ satisfying the single condition that no (-1) -dimensional simplices should be members of the subset tX . In other words, this is no more than a stratified set X together with a chosen *augmentation*, that being a delegated subset of thin 0-simplices $tX_0 \subseteq X_0$, a set of (-1) -simplices X_{-1} and a function $d_0: X_0 \rightarrow X_{-1}$ satisfying the simplicial identity $d_0 \circ d_0 = d_0 \circ d_1: X_1 \rightarrow X_{-1}$.

All of the basic definitions and results of the theory of stratified sets carry over to this context, in particular an (augmented) stratified map between such structures is simply an (augmented) simplicial map which preserves thinness. We let Strat_+ denote the category of augmented stratified sets and their stratified maps.

Observation 32. The canonical functor $: \text{Strat}_+ \rightarrow \text{Strat}$ which forgets augmentations has both a left and a right adjoint, providing us with two “opposed” augmentations of any stratified set X :

- The *canonical augmentation* (left adjoint) with $tX_0 = \emptyset$ and $X_{-1} \stackrel{\text{def}}{=} \pi_0(X)$ constructed by coequalising the pair of maps $d_0, d_1: X_1 \rightarrow X_0$.
- The *trivial augmentation* (right adjoint) with $tX_0 = X_0$ and $X_{-1} = \{*\}$ the one point set.

We make no particular choice of default augmentation, preferring instead to specify an appropriate augmentation on a case by case basis.

Definition 33 (*Joins of augmented stratified sets*). Day’s convolution construction [6] allows us to extend the monoidal structure \oplus on Δ_+ to a monoidal closed structure on Simp_+ . The tensor product of this structure, also denoted by \oplus , is called the *simplicial join* and the corresponding closures $\text{dec}_l(X, Z)$ and $\text{dec}_r(Y, Z)$ are called the *left and right décalage* constructions respectively. In line with traditional usage we will sometimes use the notations $\text{dec}_l(Z)$ and $\text{dec}_r(Z)$ to denote the décalages $\text{dec}_l(\Delta[0], Z)$ and $\text{dec}_r(\Delta[0], Z)$ respectively.

If X and Y are two (augmented) simplicial sets then Day’s convolution formula tells us that their join is given by the coend formula:

$$(X \oplus Y)_r = \int^{[n], [m] \in \Delta_+} X_n \times Y_m \times \Delta_+([r], [n] \oplus [m]).$$

A routine calculation demonstrates that an r -simplex of this join corresponds to a pair $\langle x, y \rangle$ with $x \in X_n$ and $y \in Y_m$ for some pair of integers $n, m \geq -1$ with $[n] \oplus [m] = [r]$. Under this representation if $\beta : [n'] \rightarrow [n]$ and $\gamma : [m'] \rightarrow [m]$ are simplicial operators then we have $\langle x, y \rangle \cdot (\beta \oplus \gamma) = \langle x \cdot \beta, y \cdot \gamma \rangle$ and this identity completely determines the action of Δ_+ on $X \oplus Y$ since any operator $\alpha : [r'] \rightarrow [r]$ with $[r] = [n] \oplus [m]$ may be decomposed as $\alpha = \beta \oplus \gamma$ for a unique pair of such operators.

We now extend this to (augmented) stratified sets $X, Y \in \text{Strat}_+$ by applying \oplus to their underlying (augmented) simplicial sets and letting $\langle x, y \rangle$ be thin in $X \oplus Y$ if and only if x is thin in X or y is thin in Y . It is clear that this provides a monoidal structure on Strat_+ and it is a routine matter to check that each of the endo-functors $X \oplus -$ and $- \oplus Y$ preserve the colimits of Strat_+ simply by observing that by definition they do so on underlying (augmented) simplicial sets and checking that the resulting comparison isomorphisms reflect thinness appropriately. It follows therefore that these functors have right adjoints, which we again denote by $\text{dec}_l(X, *)$ and $\text{dec}_r(Y, *)$ respectively.

Observation 34 (*Joins, décalage and augmentation*). We must, of course, augment all stratified sets before applying the join or décalage constructions to them. We shall adopt different implicit augmentation conventions for each of these operations:

Joins we apply canonical augmentation in either variable. This ensures that joins preserve colimits of stratified sets independently in each variable and that the join of two stratified sets is again canonically augmented.

Décalages we apply canonical augmentation in the first (contravariant) variable and trivial augmentation in the second (covariant) variable. This ensures that décalages carry colimits of stratified sets in the contravariant variable and limits of stratified sets in the covariant variable to limits in Strat_+ .

Observation 35 (*Augmented standard simplices and their boundaries*). In the context of augmented simplicial sets the notation $\Delta[n]$ will stand for the representable on the object $[n]$ as an object of Δ_+ and $\partial\Delta[n]$ will stand for its subset of non-surjective operators. These are all trivial augmentations of the corresponding un-augmented structures, and in most cases they coincide with the corresponding canonical augmentation. Indeed, the only exceptions to this rule are the sets $\Delta[-1]$, $\partial\Delta[0]$ and $\partial\Delta[1]$.

Observation 36 (*Joins and alternating duals*). Joins of (augmented) stratified sets are well behaved with respect to the alternating dual of Observation 26. To be precise, observe that if α and β are a pair of simplicial operators then we have $(\alpha \oplus \beta)^\circ = \beta^\circ \oplus \alpha^\circ$ from which it follows immediately that the “swap” function, which carries a pair $\langle x, y \rangle$ to the reversed pair $\langle y, x \rangle$, provides us with a stratified isomorphism between $(X \oplus Y)^\circ$ and $Y^\circ \oplus X^\circ$ which is natural in X and Y . By adjointness, these isomorphisms provide us with canonical natural isomorphisms $\text{dec}_l(X, Z)^\circ \cong \text{dec}_r(X^\circ, Z^\circ)$ and $\text{dec}_r(Y, Z)^\circ \cong \text{dec}_l(Y^\circ, Z^\circ)$.

It follows that in the sequel it will be enough to consider left joins $X \oplus -$ and the corresponding left décalage closures $\text{dec}_l(X, *)$, since the properties of right joins and décalage follow on applying alternating duals and the isomorphisms of the last paragraph.

3.2. Décalage and weak compliciality

Observation 37 (*Corner joins*). Applying the construction of Recollection 120 to the join of augmented stratified sets we obtain the *corner join* and *corner décalage* constructions which we denote by \oplus_c , dec_c^l and dec_c^r respectively. Generally we are interested in taking the corner join of

two (augmented) stratified subset inclusions $U \xrightarrow{\subseteq_s} V$ and $X \xrightarrow{\subseteq_s} Y$. In which case we know that $U \oplus Y$ and $V \oplus X$ are stratified subsets of $V \oplus Y$ with $(U \oplus Y) \cap (V \oplus X) = U \oplus X$ and it follows, by Observation 24, that we have a glueing square

$$\begin{array}{ccc} U \oplus X & \xrightarrow{\subseteq_s} & V \oplus X \\ \subseteq_s \downarrow & & \downarrow \subseteq_s \\ U \oplus Y & \xrightarrow{\subseteq_s} & (U \oplus Y) \cup (V \oplus X) \xrightarrow{\subseteq_s} V \oplus Y \end{array}$$

which demonstrates that the inclusion to the right of its lower right-hand corner is (isomorphic to) the corner join of our inclusions. One useful observation that follows from this is that if the inclusion of U into V is actually entire (cf. Definition 9, p. 1087) then $V \oplus Y$ and $U \oplus Y$ have the same underlying (augmented) simplicial sets from which it follows that the corner join depicted above is also an entire inclusion.

Observation 38 (*Joins and anodyne extensions*). By construction the join operation on Strat_+ extends ordinal sum on Δ_+ so it follows that the ordinal sum of operators provides a canonical isomorphism $\Delta[n] \oplus \Delta[m] \cong \Delta[n+m+1]$ in Strat_+ which maps each pair $\langle \alpha, \beta \rangle \in \Delta[n] \oplus \Delta[m]$ to $\alpha \oplus \beta \in \Delta[n+m+1]$. Furthermore if $0 \leq k < n$ then a simplex α is thin in the complicial simplex $\Delta^k[n]$ if and only if $\alpha \oplus \beta$ is thin in $\Delta^k[n+m+1]$ for each simplex β in $\Delta[m]$, so it follows that the isomorphism of the last sentence extends to a stratified isomorphism $\Delta^k[n] \oplus \Delta[m] \cong \Delta^k[n+m+1]$.

Now observe that if $\alpha \in \Delta[n]$ and $\beta \in \Delta[m]$ then $\alpha \oplus \beta \in \Delta^k[n+m+1]$ is in the complicial horn $\Lambda^k[n+m+1]$ if and only if α is in $\Lambda^k[n]$ or β is in $\partial \Delta[m]$. So the isomorphism of the last paragraph restricts to provide an isomorphism between the inclusion $(\Lambda^k[n] \oplus \Delta[m]) \cup (\Delta^k[n] \oplus \partial \Delta[m]) \xrightarrow{\subseteq_s} \Delta^k[n] \oplus \Delta[m]$, which is simply the corner join of the inclusions $\Lambda^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ and $\partial \Delta[m] \xrightarrow{\subseteq_r} \Delta[m]$ by Observation 37, and the complicial horn $\Lambda^k[n+m+1] \xrightarrow{\subseteq_r} \Delta^k[n+m+1]$.

This argument does not quite apply when $k = n$ for then it is not the case that $\Delta^n[n] \oplus \Delta[m]$ is isomorphic to $\Delta^n[n + m + 1]$ since they then have slightly different stratifications. However, it may be adapted to show that in this case the corner join of the last paragraph can be presented as the lower horizontal map in a pushout

$$\begin{array}{ccc} \Delta^n[n + m + 1] & \xrightarrow{\subseteq_r} & \Delta^n[n + m + 1] \\ \downarrow & & \downarrow \\ (\Delta^n[n] \oplus \Delta[m]) \cup (\Delta^n[n] \oplus \partial \Delta[m]) & \xrightarrow{\subseteq_r} & \Delta^n[n] \oplus \Delta[m] \end{array} \quad (1)$$

and is thus an anodyne extension (cf. Definition 22, p. 1091).

We may apply a similar argument to the corner join of $\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ and $\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_t$, which is an entire subset inclusion since the second of these inclusions is itself entire (Observation 37). Now, if a non-degenerate simplex $\langle \alpha, \beta \rangle$ is thin in the codomain $\Delta^k[n] \oplus \Delta[m]_t$ of this corner join then by definition either α is thin in $\Delta^k[n]$, in which case $\langle \alpha, \beta \rangle$ is thin in $\Delta^k[n] \oplus \Delta[m]$, or β is thin in $\Delta[m]_t$, in which case it is also thin in $\Delta^k[n] \oplus \Delta[m]_t$ unless α fails to be a simplex of the horn $\Delta^k[n]$ altogether. In that latter case either $\alpha = \text{id}_{[n]}$ or $\alpha = \delta_k$, and again the first of these would make $\langle \alpha, \beta \rangle$ thin in $\Delta^k[n] \oplus \Delta[m]$. Summarising this argument, we see that if the simplex $\langle \alpha, \beta \rangle$ is thin in the codomain of the corner join under consideration and non-thin in its domain then it can only be the simplex $\langle \delta_k, \text{id}_{[m]} \rangle \in \Delta[n] \oplus \Delta[m]$, which corresponds to the simplex $\delta_k \in \Delta[n + m + 1]$ under the canonical ordinal sum isomorphism $\Delta[n] \oplus \Delta[m] \cong \Delta[n + m + 1]$. It follows, immediately, that we may present this corner join as the lower horizontal of a pushout square

$$\begin{array}{ccc} \Delta^k[n + m + 1]' & \xrightarrow{\subseteq_r} & \Delta^k[n + m + 1]'' \\ \downarrow & & \downarrow \\ (\Delta^k[n] \oplus \Delta[m]_t) \cup (\Delta^k[n] \oplus \Delta[m]) & \xrightarrow{\subseteq_r} & \Delta^k[n] \oplus \Delta[m]_t \end{array}$$

and we may infer that it is thus an inner anodyne extension. Finally, an analogous analysis of the thinness extension $\Delta^k[n]' \xrightarrow{\subseteq_e} \Delta^k[n]''$ shows that its corner joins with the boundary and thin simplex inclusions can again be obtained as pushouts of the thinness extension $\Delta^k[n + m + 1]' \xrightarrow{\subseteq_e} \Delta^k[n + m + 1]''$, which again demonstrates that they are both anodyne extensions.

Summarising these observations we get the following lemma and its obvious dual involving left-handed décalage:

Lemma 39 (Weak complicial sets and décalage). *If $e: U \hookrightarrow V$ is an anodyne extension and $i: X \hookrightarrow Y$ is any inclusion (monomorphism) of augmented stratified sets then their corner join $e \oplus_c i$ is an anodyne extension. It follows that if $p: E \rightarrow B$ is a complicial fibration (cf. Definition 22, p. 1091) then so is the right corner décalage $\text{dec}_r^c(i, p)$.*

Thus, if Y is any augmented stratified set then the endo-functor $- \oplus Y$ preserves all anodyne extensions and if A is a weak complicial set then so is $\text{dec}_r(Y, A)$.

Proof. To prove the first part, observe that we may extend the argument of Example 21 to show that the class of all inclusions of augmented stratified sets is the cellular completion (cf. Notation 20, p. 1091) of the set of all boundary and thin simplex inclusions:

$$\{\partial\Delta[n] \xrightarrow{\subseteq_r} \Delta[n] \mid n = -1, 0, 1, \dots\} \cup \{\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_t \mid n = 0, 1, 2, \dots\}. \quad (2)$$

The calculations of the last observation demonstrated that the corner join of any of the inclusions in this set with an elementary anodyne extension is again an anodyne extension, so we may apply Lemma 121 to extend this result to all inclusions and anodyne extensions as required. The second sentence of the statement now follows by applying Observation 122.

Finally, observe that if $\emptyset \xrightarrow{i} Y$ is the unique inclusion from the empty augmented stratified set into Y then the corner join $e \oplus_c i$ is clearly isomorphic to $e \oplus Y : U \oplus Y \rightarrow V \oplus Y$, since joins preserve the initial object \emptyset which implies that the domain $(V \oplus \emptyset) \vee_{U \oplus \emptyset} (U \oplus Y)$ of our corner join is isomorphic to $U \oplus Y$. Similarly the corner décalage $\text{dec}_r^c(i, p)$ is isomorphic to the map $\text{dec}_r(Y, p) : \text{dec}_r(Y, E) \rightarrow \text{dec}_r(Y, B)$. It follows that the result of the last paragraph specialises to establish the final sentence of the statement. \square

Observation 40 (*Inner anodyne extensions and joins*). Notice that Observation 38 actually demonstrates that if we corner join an elementary **inner** anodyne extension with a boundary or thin simplex inclusion then the resulting map is in fact also an inner anodyne extension. This immediately implies that Lemma 39 has a direct analogue in which anodyne extensions, complicial fibrations and weak complicial sets are replaced by their inner counterparts.

However, consulting Observation 38 again in greater detail we see that a little more is true. In particular, observe that in the pushout of display (1) the upper horizontal map is actually an inner horn extension whenever $m \geq 0$ and so it follows that its lower horizontal, the corner join

of the right outer horn $\Lambda^n[n] \xrightarrow{\subseteq_r} \Delta^n[n]$ and the boundary inclusion $\partial\Delta[m] \xrightarrow{\subseteq_r} \Delta[m]$, is also an inner anodyne extension. A similar comment holds for the other three cases of Observation 38 in which an elementary right outer anodyne extension is corner joined with a boundary or thin simplex inclusion. It follows therefore, by applying Lemma 121 again, that the corner join $e \oplus_c i$ of a **right** anodyne extension e and an inclusion i in the cellular completion of the set obtained by

removing $\partial\Delta[-1] \xrightarrow{\subseteq_r} \Delta[-1]$ from the set in display (2) is actually an **inner** anodyne extension. Furthermore, a simple argument demonstrates that a map i in this latter cellular completion if and only if it is an inclusion of augmented stratified sets which acts isomorphically on sets of (-1) -simplices.

4. Equivalences in weak complicial sets

In this section we provide an alternative characterisation of weak complicial sets, which replaces outer complicial horn fillers with an *equivalence* condition on thin 1-simplices. This theory directly generalises the analysis of quasi-isomorphisms given by Joyal in his paper on quasi-categories [11], which itself is implicit in Vogt's work on homotopy limits [25] and Cordier and Porter's work on homotopy coherence [3] and [4]. Herein the material of this section primarily serves to simplify subsequent work, by freeing us from directly analysing certain special cases involving outer horns.

4.1. Equivalences in simplicial sets

Definition 41 (*The generic simplicial equivalence*). Let \mathbb{I} denote the chaotic category on two objects $\{-, +\}$, which is generally referred to as the *generic isomorphism*, and let $E \in \underline{\text{Simp}}$ denote its nerve (cf. Observation 4, p. 1086). In other words, E is the simplicial set whose m -simplices are, not necessarily order preserving, functions $e: [m] \rightarrow \{-, +\}$ upon which simplicial operators act by pre-composition. We can think of an m -simplex of E as a sequence $e_0 e_1 \dots e_m$ of the symbols $-$ and $+$ of length $n + 1$ upon which a simplicial operator $\alpha: [n] \rightarrow [m]$ acts by re-indexing $(e_0 e_1 \dots e_m) \cdot \alpha = e_{\alpha(0)} e_{\alpha(1)} \dots e_{\alpha(n)}$. For reasons that will become apparent, we call E the *generic simplicial equivalence* and we say that a 1-simplex v of a simplicial set X is a (*simplicial*) *equivalence* if there exists some simplicial map $f: E \rightarrow X$ with $f(-+) = v$.

In what follows, we sometimes use the symbols p and q to represent elements of $\{-, +\}$ and use the notation \neg to denote the function which swaps $+$ and $-$.

Observation 42 (*Decomposing E*). An n -simplex $e \in E$ is degenerate iff there is some $i \in [n - 1]$ for which $e_i = e_{i+1}$. It follows that E has exactly 2 non-degenerate n -simplices, these being the two alternating sequences of length $n + 1$ starting from $-$ and $+$ respectively for which we reserve the notation $e_n^- = - + - + \dots$ and $e_n^+ = + - + - \dots$.

Let E_n^p denote the simplicial subset of E generated by the simplex e_n^p and observe that the obvious identities $e_{n+1}^p \cdot \delta_0 = e_n^{-p}$ and $e_{n+1}^p \cdot \delta_{n+1} = e_n^p$ imply that we have $E_n^q \subseteq_s E_{n+1}^p$ for each $n \in \mathbb{N}$ and $p, q \in \{-, +\}$ and that E itself is the union of the increasing chain $E_1^p \subseteq_s E_2^p \subseteq_s \dots \subseteq_s E_n^p \subseteq_s \dots$. Furthermore, they also imply that the only two non-degenerate simplices of E_{n+1}^p which are not in E_n^p (resp. E_n^{-p}) are e_{n+1}^p itself and its face $e_n^{-p} = e_{n+1}^p \cdot \delta_0$ (resp. $e_n^p = e_{n+1}^p \cdot \delta_{n+1}$). It follows that we have canonical pushout squares

$$\begin{array}{ccc} \Lambda^0[n+1] & \xrightarrow{\subseteq_s} & \Delta[n+1] \\ \downarrow & \lrcorner & \downarrow \lceil e_{n+1}^p \rceil \\ E_n^p & \xrightarrow{\subseteq_s} & E_{n+1}^p \end{array} \quad \begin{array}{ccc} \Lambda^{n+1}[n+1] & \xrightarrow{\subseteq_s} & \Delta[n+1] \\ \downarrow & \lrcorner & \downarrow \lceil e_{n+1}^{-p} \rceil \\ E_n^{-p} & \xrightarrow{\subseteq_s} & E_{n+1}^p \end{array} \quad (3)$$

in $\underline{\text{Simp}}$, in which $\lceil e_{n+1}^p \rceil: \Delta[n+1] \rightarrow E_{n+1}^p$ is the simplicial map that corresponds to the $(n+1)$ -simplex $e_{n+1}^p \in E_{n+1}^p$ via Yoneda's lemma. We will also use the notation E_n to denote the union of the subsets E_n^- and E_n^+ in E .

Observation 43 (*Equivalences in weak complicial sets*). From hereon we will adopt the (slightly nonstandard) convention that the simplicial sets E_n^- , E_n^+ and E_n are all stratified with the 0-trivialised stratification, in which a simplex is thin iff its dimension is greater than 0. To recover the default minimal stratification on these sets we apply the underlying simplicial set notation \tilde{E}_n^- , \tilde{E}_n^+ and \tilde{E}_n of Definition 5 and appeal to the default stratification rule of Observation 6.

Lifting the left-hand pushout of display (3) to $\underline{\text{Strat}}$ we find that the inclusion $E_n^p \xrightarrow{\subseteq_r} E_{n+1}^p$ is a pushout of the left horn extension $\Lambda^0[n+1]' \xrightarrow{\subseteq_r} \Delta^0[n+1]'$ and is thus itself a left anodyne extension. Arguing dually we see that the inclusion $E_n^{-p} \xrightarrow{\subseteq_r} E_{n+1}^p$ is a right anodyne extension.

Taking composites of these it follows that each inclusion $E_m^p \xrightarrow{\subseteq_r} E_n^p$ is a left anodyne extension and that $E_m^q \xrightarrow{\subseteq_r} E_n^p$ is a right anodyne extension if $(n - m)$ is even and $p = q$ or if $(n - m)$ is odd and $p = \neg q$.

In particular, the inclusion $E_1^- \xrightarrow{\subseteq_r} E$ may be constructed as a countable composite of the inclusions $E_1^- \subseteq_r E_2^- \subseteq_r E_3^- \subseteq_r \dots$ so it follows from the last paragraph that it is a left anodyne extension. Indeed, we may also construct it as a countable composite of the (alternating) sequence $E_1^- \subseteq_r E_2^+ \subseteq_r E_3^- \subseteq_r E_4^+ \subseteq_r \dots$ which demonstrates that it is also a right anodyne extension.

Now observe that the stratified set E_1^- is simply isomorphic to the standard thin 1-simplex $\Delta[1]_t$, so it follows that any thin 1-simplex v of a weak complicial set A gives rise to a unique stratified map $\ulcorner v \urcorner: E_1^- \rightarrow A$ with $\ulcorner v \urcorner(e_1^-) = v$. Since A is a weak complicial set this may be lifted along the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E$ to give a stratified map which demonstrates that v is a equivalence in the underlying simplicial set of A .

Observation 44 (*Some symmetries of E*). In the sequel we will have use for a couple of canonical isomorphisms defined upon E :

Symmetry (i). The function $\neg: E \rightarrow E$ which applies the parity swapping function \neg pointwise to the symbols comprising each simplex of E and is clearly an idempotent map of simplicial sets. Furthermore, this restricts to provide an isomorphism between E_n^p and E_n^{-p} for each $n \in \mathbb{N}$ and $p \in \{-, +\}$.

Symmetry (ii). The function “rev” which reverses the order of the symbols in each simplex of E and is clearly the underlying function of a mutually inverse pair of simplicial isomorphisms $\text{rev}: E \rightarrow E^\circ$ and $\text{rev}: E^\circ \rightarrow E$. Furthermore these restrict to provide an isomorphism between $(E_n^p)^\circ$ and E_n^p if n is even and E_n^{-p} if n is odd.

4.2. Equivalences and inner compliciality

Conversely, the following sequence of observations and lemmas demonstrate that an equivalence property on thin 1-simplices is enough to ensure that a weak inner complicial set has outer horn fillers. This result may be considered to be a complicial generalisation of Joyal’s analysis of *special horn fillers* in quasi-categories [11].

Observation 45. Our primary goal over the next few lemmas will be to show that a weak inner complicial set A is actually a weak complicial set iff it has the RLP (cf. Notation 14, p. 1089) with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$.

To that end we start by observing that the inclusion $E_0^- \xrightarrow{\subseteq_r} E_1^-$ is isomorphic to $\Delta(\varepsilon_0): \Delta[0] \xrightarrow{\subseteq_r} \Delta[1]_t$ and arguing as in Observation 38 to show that its corner join with the inclusion $\partial\Delta[m] \xrightarrow{\subseteq_r} \Delta[m]$ is isomorphic to the left outer horn $\Lambda^0[m+2] \xrightarrow{\subseteq_r} \Delta^0[m+2]$. This leads us to considering the following increasing sequence of stratified subsets of $E_3^- \oplus \Delta[m]$ which starts with the domain of this corner join

$$\begin{aligned}
X_0 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]) \cup (E_1^- \oplus \partial \Delta[m]), \\
X_1 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]) \cup (E_1^- \oplus \partial \Delta[m]) \cup (E_3^- \oplus \{*\}), \\
X_2 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]) \cup (E_3^- \oplus \partial \Delta[m]), \\
X_3 &\stackrel{\text{def}}{=} (E_2^- \oplus \Delta[m]) \cup (E_3^- \oplus \partial \Delta[m])
\end{aligned}$$

and ends with a stratified set containing its codomain $E_1^- \oplus \Delta[m]$. Here $\{*\}$ in the definition of X_1 represents the stratified subset of $\Delta[m]$ containing only its unique (-1) -dimensional simplex $\iota^m : [-1] \rightarrow [m]$. The following observations follow directly from these definitions:

Observation (i). We have $X_0 \cup (E_3^- \oplus \{*\}) = X_1$ and $X_0 \cap (E_3^- \oplus \{*\}) = E_1^- \oplus \{*\}$ so we obtain a glueing square which presents the inclusion $X_0 \xrightarrow{\subseteq_r} X_1$ as a pushout of the inclusion $E_1^- \oplus \{*\} \xrightarrow{\subseteq_r} E_3^- \oplus \{*\}$. Furthermore this, in turn, is isomorphic to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$ of the statement, since $\{*\}$ is isomorphic to $\Delta[-1]$ the identity for \oplus .

Observation (ii). We have the equalities $X_1 \cup (E_3^- \oplus \partial \Delta[m]) = X_2$ and $X_1 \cap (E_3^- \oplus \partial \Delta[m]) = (E_1^- \oplus \partial \Delta[m]) \cup (E_3^- \oplus \{*\})$ thus ensuring that we have a glueing square which presents the inclusion $X_1 \xrightarrow{\subseteq_r} X_2$ as a pushout of the inclusion $(E_1^- \oplus \partial \Delta[m]) \cup (E_3^- \oplus \{*\}) \xrightarrow{\subseteq_r} E_3^- \oplus \partial \Delta[m]$ which, in turn, is the corner join of $E_1^- \xrightarrow{\subseteq_r} E_3^-$ and $\{*\} \xrightarrow{\subseteq_r} \partial \Delta[m]$ (by Observation 37). Now the first of these is a right anodyne extension, as demonstrated in Observation 43, so we may apply Observation 40 to show that their corner join is an inner anodyne extension.

Observation (iii). We have the equalities $X_2 \cup (E_2^- \oplus \Delta[m]) = X_3$ and $X_2 \cap (E_2^- \oplus \Delta[m]) = (E_0^- \oplus \Delta[m]) \cup (E_2^- \oplus \partial \Delta[m])$ thus ensuring that we have a glueing square which presents the inclusion $X_2 \xrightarrow{\subseteq_r} X_3$ as a pushout of the inclusion $(E_0^- \oplus \Delta[m]) \cup (E_2^- \oplus \partial \Delta[m]) \xrightarrow{\subseteq_r} E_2^- \oplus \Delta[m]$ which, in turn, is the corner join of $E_0^- \xrightarrow{\subseteq_r} E_2^-$ and $\partial \Delta[m] \xrightarrow{\subseteq_r} \Delta[m]$ (by Observation 37). Now the first of these is a right anodyne extension, as demonstrated in Observation 43, so we may apply Observation 40 to show that their corner join is an inner anodyne extension.

From these it follows immediately that the inclusion $X_0 \xrightarrow{\subseteq_r} X_3$ is in the cellular completion of the set of inclusions obtained by adding $E_1^- \xrightarrow{\subseteq_r} E_3^-$ to the set of elementary inner anodyne extensions.

Observation 46. A similar sequence of observations hold for complicial thinness extensions, however this time we consider the corner join of $E_0^- \xrightarrow{\subseteq_r} E_1^-$ and $\Delta[m] \xrightarrow{\subseteq_e} \Delta[m]_t$ and argue along the lines presented in the latter part of Observation 38 to show that it is isomorphic to the left outer thinness extension $\Delta^0[m+2]' \xrightarrow{\subseteq_e} \Delta^0[m+2]''$. This again leads us to considering an increasing sequence of stratified subsets of $E_3^- \oplus \Delta[m]_t$ which starts with the domain of our corner join

$$\begin{aligned}
Y_0 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]_t) \cup (E_1^- \oplus \Delta[m]), \\
Y_1 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]_t) \cup (E_1^- \oplus \Delta[m]) \cup (E_3^- \oplus \{*\}), \\
Y_2 &\stackrel{\text{def}}{=} (E_0^- \oplus \Delta[m]_t) \cup (E_3^- \oplus \Delta[m]), \\
Y_3 &\stackrel{\text{def}}{=} (E_2^- \oplus \Delta[m]_t) \cup (E_3^- \oplus \Delta[m])
\end{aligned}$$

and ends with a stratified set containing its codomain $E_1^- \oplus \Delta[m]_t$. Arguing exactly as we did in the subclauses of the last observation, we see that the first inclusion in this sequence is a pushout of $E_1^- \xrightarrow{\subseteq_r} E_3^-$ and that its last two are both inner anodyne extensions.

Lemma 47 (*Lifting of equivalences is enough*). Suppose that B is a weak complicial set and $p: A \rightarrow B$ is an inner complicial fibration (cf. Definition 22, p. 1091) which has the RLP with respect to the inclusions $E_1^- \xrightarrow{\subseteq_r} E_3^-$ and $E_0^- \xrightarrow{\subseteq_r} E_1^-$ then p is a complicial fibration. Consequently, A is a weak complicial set iff it is a weak inner complicial set which has the RLP with respect to $E_1^- \xrightarrow{\subseteq_r} E_3^-$.

Proof. First observe that we may apply (both of) the symmetry isomorphisms of Observation 44 to show that the dual inclusion $(E_1^-)^\circ \xrightarrow{\subseteq_r} (E_3^-)^\circ$ is actually isomorphic to $E_1^- \xrightarrow{\subseteq_r} E_3^-$ itself. On the other hand the inclusion $E_0^- \xrightarrow{\subseteq_r} E_1^-$ is not self dual, instead its dual is isomorphic to $E_0^- \xrightarrow{\subseteq_r} E_1^+$. However, it is still the case that the conditions of the statement are enough to ensure that $p: A \rightarrow B$ also has the RLP with respect to this latter inclusion.

To see that this is the case, simply observe that E_1^+ is also a stratified subset of E_3^- from which it follows that we may solve a lifting problem (u, v) in the following diagram

$$\begin{array}{ccc}
E_0^- & \xrightarrow{u} & A \\
\downarrow \subseteq_r & \nearrow \subseteq_r & \downarrow p \\
& E_3^- & \\
E_1^+ & \xrightarrow{i} & B \\
& \searrow v & \\
& &
\end{array}$$

(Note: The diagram also includes dotted arrows $w: A \rightarrow E_3^-$ and $v': E_3^- \rightarrow B$ forming a triangle with i .)

in two steps. First use the weak compliciality of B to extend the stratified map v along the anodyne extension $E_1^+ \xrightarrow{\subseteq_r} E_3^-$ (cf. Observation 43, p. 1101) to obtain the dotted map v' . Then

observe that the inclusion $E_0^- \xrightarrow{\subseteq_r} E_3^-$ may be decomposed as a composite of the inclusions identified in the statement, from which it follows that p also has the RLP with respect this latter inclusion. This allows us to solve the new lifting problem (u, v') and obtain the stratified map w , which we compose with the inclusion i to finally construct the desired solution to the original lifting problem.

Applying this result and the fact that the class of inner complicial fibrations is closed under alternating duals (cf. Observation 26, p. 1093), we have demonstrated that $p: A \rightarrow B$ satisfies the conditions given in the statement if and only if its alternating dual $p^\circ: A^\circ \rightarrow B^\circ$ satisfies

them. Consequently, to demonstrate that $p : A \rightarrow B$ is a complicial fibration it is enough to show that it has the RLP with respect to all left outer horns and thinness extensions, because then we may demonstrate the corresponding right-handed result for p by appealing to the already established left handed one for $p^\circ : A^\circ \rightarrow B^\circ$.

Now to prove that p is a left complicial fibration, first observe that the inclusions $E_0^- \xrightarrow{\subseteq_r} E_1^-$ and $\Delta^0[1] \xrightarrow{\subseteq_r} \Delta^1[1]$ are isomorphic and so the statement already assumes left outer horn lifting at dimension 1. At dimension 2 and above apply Observation 45 to replace the left outer horn $\Delta^0[m+2] \xrightarrow{\subseteq_r} \Delta^0[m+2]$ ($m = 0, 1, 2, \dots$) by the isomorphic inclusion $X_0 \xrightarrow{\subseteq_r} E_1^- \oplus \Delta[m]$ and then seek to solve the lifting problems (u, v) of the form in the diagram

$$\begin{array}{ccc}
 X_0 & \xrightarrow{u} & A \\
 \downarrow \subseteq_r & \nearrow w & \downarrow p \\
 & X_3 & \\
 & \searrow j & \\
 E_1^- \oplus \Delta[m] & \xrightarrow{v} & B \\
 & \nearrow i & \\
 & E_3^- \oplus \Delta[m] & \\
 & \nearrow v' &
 \end{array}
 \quad (4)$$

in a couple of steps. First we use the weak compliciality of B to construct the map v' by extending the stratified map v along the regular inclusion $E_1^- \oplus \Delta[m] \xrightarrow{\subseteq_r} E_3^- \oplus \Delta[m]$, which is an anodyne extension as it may be constructed by applying the anodyne extension preserving right join functor $- \oplus \Delta[m]$ (cf. Observation 39, p. 1099) to the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E_3^-$ (cf. Observation 43, p. 1101). Now we again consult Observation 45 to see that the inclusion $X_0 \xrightarrow{\subseteq_r} X_3$ is in the cellular completion of the set consisting of the elementary inner anodyne extensions and the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$ of the statement, so in particular it follows that the assumed injectivity properties of p imply that it has the RLP with respect to this inclusion. Using this fact we may solve the new lifting problem $(u, v' \circ j)$ and obtain the stratified map w , which we compose with the inclusion i to finally construct the desired solution to the original lifting problem.

An identical argument which, this time, uses the results described in Observation 46 demonstrates that p also has the RLP with respect to each elementary thinness extension $\Delta^0[m+2]' \xrightarrow{\subseteq_e} \Delta[m+2]''$ ($m = 0, 1, 2, \dots$). This completes our proof that p is a left complicial fibration and finally establishes the first sentence of the statement, by applying our comments on duality above. The second sentence of the statement follows from the first simply by observing that 1 is a weak complicial set and thus that A satisfies the conditions of the latter iff the unique map $p : A \rightarrow 1$ satisfies the conditions of the former. \square

Observation 48. Notice that in the last lemma we did not need to explicitly assume that A had the RLP with respect to the inclusion $E_0^- \xrightarrow{\subseteq_r} E_1^-$ in order for it to be a weak complicial set. Indeed, it is the case that all stratified sets have this property, since the unique stratified map from

E_1^- to $E_0^- \cong \Delta[0]$ is (trivially) left inverse to the $E_0^- \xrightarrow{\subseteq_r} E_1^-$ and may thus be composed with any lifting problem $E_0^- \rightarrow X$ to construct its solution.

Corollary 49 (*Lifting of left or right outer horn fillers is enough*). Suppose that B is a weak complicial set and $p: A \rightarrow B$ is a left complicial fibration then p is a complicial fibration. Consequently, A is a weak complicial set iff it is a weak left complicial set. Applying these results to the alternating duals $p^\circ: A^\circ \rightarrow B^\circ$ and A° we also obtain the corresponding results for right compliciality.

Proof. By Observation 43 the inclusions $E_1^- \xrightarrow{\subseteq_r} E_3^-$ and $E_0^- \xrightarrow{\subseteq_r} E_1^-$ are both left anodyne extensions, so the assumption that $p: A \rightarrow B$ is a left complicial fibration implies that it has the RLP with respect to those inclusions and thus satisfies the conditions of the last lemma. \square

Corollary 50. If B is a weak complicial set and $p: A \rightarrow B$ is an inner complicial fibration then any lifting problem

$$\begin{array}{ccc} \Lambda^0[m+2] & \xrightarrow{u} & A \\ \subseteq_r \downarrow & & \downarrow p \\ \Delta^0[m+2] & \xrightarrow{v} & B \end{array} \quad (5)$$

($m \geq 0$) has a solution so long as u maps the 1-simplex with vertices 0 and 1 to a degenerate 1-simplex in A .

Proof. Simply a minor modification of that part of the proof of Lemma 47 surrounding display (4), the details of which we leave to the reader. \square

4.3. Equivalence inverses

In this subsection we refine Lemma 47 one step further.

Observation 51 (*Illustrating low-dimensional calculations*). In some of what follows, it will be useful to illustrate certain low-dimensional calculations in our stratified sets. To do so we resort to drawing simplices as *oriental diagrams*, which were introduced by Street in [20]. It should be noted, however, that for us this is simply a convenient way of drawing simplices on the 2-dimensional page rather than a way of describing them as free (strict) ω -categories.

For example, doing so immediately illuminates the meaning of the RLP with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$ which was so central to the work of the last subsection. Diagrammatically it states that for each thin 1-simplex $v \in tA_1$ there exists some 3-simplex $t = \hat{v}(e_3^-) \in A$ which may be pictured as:

Here we adopt the diagrammatic convention of labelling degenerate simplices using the equality symbol $=$ and thin simplices with the equivalence symbol \simeq . When drawn in this form the intention of our definition immediately becomes plain, viz “ v has equivalence inverse w and the associated thin 2-simplices $\text{id}_x \simeq w \circ v$ and $\text{id}_y \simeq v \circ w$ have been chosen to satisfy a certain 3-cocycle condition.”

Definition 52. Let E'_2 be the stratified set $E_2^- \vee_{E_1^+} E_2^+$, that is to say let it be constructed by forming the pushout:

Of course, each of the inclusions i_0 and i_1 is an anodyne extension since they are, by definition, pushouts of regular subset inclusions which we showed to be anodyne extensions in Observation 43. We will also use $i : E_1^- \hookrightarrow E'_2$ to denote the anodyne extension obtained by composing

$i_0 : E_2^- \hookrightarrow E'_2$ of the last sentence and the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E_2^-$ of Observation 43.

More explicitly, we may represent a stratified map $f : E'_2 \rightarrow A$ diagrammatically as a pair of thin 2-simplices

in A . In other words, this amounts to a 1-simplex v in A with a right equivalence inverse w which itself, in turn, has a right equivalence inverse v' .

Notice that the regular subset $E_2 \subseteq_r E$ is **not** isomorphic to E'_2 , a fact which follows as soon as we observe that a stratified map $f : E_2 \rightarrow A$ simply amounts to a pair of 1-simplices v and w which are mutual equivalence inverses. It is clear, therefore, that we may construct E_2 from E'_2 by taking a quotient which identifies the 1-simplices labelled with v and v' in diagram (8).

Lemma 53. Suppose B is a weak complicial set and that the inner complicial fibration $p : A \rightarrow B$ has the RLP with respect to the inclusion $i : E_1^- \hookrightarrow E'_2$ then it has the RLP with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$.

- The regular subset E'_2 includes those simplices labelled v, v', w, a and a' in Fig. 1, so in particular it includes the data for a 1-complicial horn (cf. Notation 10, p. 1087) on the vertices labelled 0, 1, 2, 4 and it is clear therefore that we may construct the inclusion $E'_2 \xrightarrow{\subseteq_r} U_1$ as a pushout of the thin inner horn $\Lambda^1[3]' \xrightarrow{\subseteq_r} \Delta^1[3]''$ along the evident stratified map which carries the vertices of its domain to those labelled 0, 1, 2, 4 in E'_2 .
- The regular subset U_1 contains the 3-simplex x and thus also contains its 2-face c so we see that this set includes the data for a 0-complicial horn on the vertices labelled 0, 2, 3, 4 and it is thus clear that we may construct $U_1 \xrightarrow{\subseteq_r} U_2$ as a pushout of the thin horn $\Lambda^0[3]' \xrightarrow{\subseteq_r} \Delta^0[3]''$ along the evident stratified map which carries the vertices of its domain to those labelled 0, 2, 3, 4 in U_1 . While this is an outer horn, the 1-simplex with vertices labelled 0 and 2 is degenerate in C and so Corollary 50 applies in this case to show that the inclusion $U_1 \xrightarrow{\subseteq_r} U_2$ does have the LLP with respect to p as required.
- The regular subset U_2 contains the 3-simplex y and its 2-face d so we see that this set includes the data for a 1-complicial horn on the vertices labelled 1, 2, 3, 4 and it is thus clear that we may construct $U_2 \xrightarrow{\subseteq_r} U_3$ as a pushout of the thin inner horn $\Lambda^1[3]' \xrightarrow{\subseteq_r} \Delta^1[3]''$ along the evident stratified map which carries the vertices of its domain to those labelled 1, 2, 3, 4 in U_2 .
- Finally the regular subset U_3 contains all of the 3-simplices labelled x, y and z and it contains the 3-simplex with vertices labelled 0, 2, 3, 4, since that is a degenerate 3-simplex with 2-face a which is in E'_2 , so we see that this set includes the data for a 4-dimensional 2-complicial horn and it is thus clear that we may construct $U_3 \xrightarrow{\subseteq_r} C$ as a pushout of the thin inner horn $\Lambda^2[4]' \xrightarrow{\subseteq_r} \Delta^2[4]''$ along the evident stratified map which carries the vertices of its domain to those labelled 0, 1, 2, 3, 4 in U_3 .

So each one of these inclusions enjoys the LLP with respect to p and it follows therefore that their composite does.

Now observe that we may construct a stratified map $r : C \rightarrow E_3^-$ which maps the vertex variously labelled 0, 2, and 3 in Fig. 1 to $-$ and the one labelled 1 and 4 to $+$. This is a retraction in the sense that if we pre-compose it with the inclusion $E_3^- \xrightarrow{\subseteq_r} C$ we obtain the identity on E_3^- . So suppose that $p : A \rightarrow B$ is an inner complicial fibration, B is a weak complicial set and that p has the RLP with respect to the inclusion $E_1^- \xrightarrow{i} E'_2$ and consider the lifting problem depicted as the outer square in:

$$\begin{array}{ccc}
 E_1^- & \xrightarrow{f} & A \\
 \downarrow \subseteq_r & \searrow i & \downarrow p \\
 & E'_2 & \\
 E_3^- & \xrightarrow{g} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \\
 & \nearrow h' & \\
 & C & \\
 & \searrow h & \\
 & &
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \\
 & \nearrow g \circ r & \\
 & &
 \end{array}$$

To solve this, we first form the composite $g \circ r$ which completes the data for a lifting problem from the inclusion $i: E_1^- \hookrightarrow E_2'$ to $p: A \rightarrow B$, because r is a retract of the inclusion from E_3^- to C and the (skewed) square of inclusions to the left of our diagram commutes. Solving this problem, which we may do by assumption on p , we obtain the map h' which in turn furnishes us with a lifting problem $(h', g \circ r)$ from $E_2' \xrightarrow{\subseteq_r} C$ to p . However the proof of the last paragraph tells us that these enjoy the lifting property with respect to each other, so we may solve this latter problem to obtain the map h and thus solve our original problem with the composite of that map and the inclusion $E_3^- \xrightarrow{\subseteq_r} C$ as required. \square

As a corollary, we find that a weak inner complicial set is actually a weak complicial set if and only if each of its thin 1-simplices has an equivalence inverse:

Corollary 54. *If A is a weak inner complicial set then it is a weak complicial set iff it has the RLP with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$.*

Proof. For the “only if” implication, if A is a weak complicial set then we may lift any stratified map $f: E_1^- \rightarrow A$ along the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E_3^-$ and compose the resulting map with the inclusion $E_2 \xrightarrow{\subseteq_r} E_3^-$ to construct a stratified map which provides the required lift of f along $E_1^- \xrightarrow{\subseteq_r} E_2$.

For the reverse implication, we know that E_2 is a quotient of E_2' and that we may decompose the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$ as the composite of the inclusion $i: E_1^- \hookrightarrow E_2'$ and the quotient map $q: E_2' \twoheadrightarrow E_2$. So if A has the lifting property of the statement then we may show that it has the RLP with respect the inclusion i by lifting along $E_1^- \xrightarrow{\subseteq_r} E_2$ and composing with q . It follows that we may apply Lemma 53 to show that A also has the RLP with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_3^-$ which, in turn, allows us to apply Lemma 47 and demonstrate that it is a weak complicial set as required. \square

Corollary 55. *If A and B are weak complicial sets and $p: A \rightarrow B$ is an inner complicial fibration then it is a complicial fibration iff it has the RLP with respect to the inclusion $E_0^- \xrightarrow{\subseteq_r} E_1^-$.*

Proof. The “only if” direction is immediate, since the cited inclusion is isomorphic to the elementary anodyne extension $\Lambda^0[1] \xrightarrow{\subseteq_r} \Delta^0[1]$. For the reverse implication we start by demonstrating that p has the RLP with respect to the inclusion $\Lambda^0[2] \xrightarrow{\subseteq_r} \Delta^0[2]$. To that end define three stratified sets

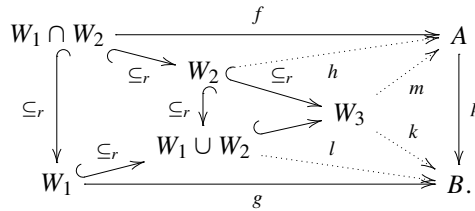
$$W_1 \stackrel{\text{def}}{=} E_1^- \oplus \Delta[0], \quad W_2 \stackrel{\text{def}}{=} (E_0^- \oplus \Delta[0]) \cup (E_2^- \oplus \partial\Delta[0]), \quad W_3 \stackrel{\text{def}}{=} E_2^- \oplus \Delta[0]$$

for which $W_1, W_2 \subseteq_r W_3$ and

$$W_1 \cap W_2 = (E_0^- \oplus \Delta[0]) \cup (E_1^- \oplus \partial\Delta[0]),$$

$$W_1 \cup W_2 = (E_1^- \oplus \Delta[0]) \cup (E_2^- \oplus \partial\Delta[0])$$

consequently, arguing as in Observation 38, we find that the horn inclusion of the last sentence is isomorphic to $W_1 \cap W_2 \xrightarrow{\subseteq_r} W_1$. To show that p has the RLP with respect to this latter inclusion consider the lifting problem (f, g) shown in the outer square of the following diagram:



Here, we construct the various dotted maps in the following sequence:

- The inclusion $W_1 \cap W_2 \xrightarrow{\subseteq_r} W_2$ may be formed by taking the pushout of the corner join of the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E_2^-$ and the inclusion $\emptyset \xrightarrow{\subseteq_r} \partial\Delta[0]$ along the inclusion $E_1^- \oplus \partial\Delta[0] \xrightarrow{\subseteq_r} W_2$. So, applying Lemma 39, we find that this inclusion is an anodyne extension and thus that we may factor the map f through W_2 , since A is a weak complicial set by assumption, to give the map labelled h .
- Now the map l may be constructed using the pushout property of the pasting square determined by W_1 and W_2 to the left of the diagram.
- The inclusion $W_1 \cup W_2 \xrightarrow{\subseteq_r} W_3$ is simply the corner join of the anodyne extension $E_1^- \xrightarrow{\subseteq_r} E_2^-$ (cf. Observation 43, p. 1101) and the boundary inclusion $\partial\Delta[0] \xrightarrow{\subseteq_r} \Delta[0]$. Applying Lemma 39, we find that this inclusion is an anodyne extension and thus that we may factor the map l through W_3 , since B is a weak complicial set by assumption, to give the map labelled k above.
- The inclusion $W_2 \xrightarrow{\subseteq_r} W_3$ may be constructed as the corner join of the right anodyne extension $E_0^- \xrightarrow{\subseteq_r} E_2^-$ (cf. Observation 43, p. 1101) and the inclusion $\partial\Delta[0] \xrightarrow{\subseteq_r} \Delta[0]$. Applying Observation 40, we find that this inclusion is an *inner* anodyne extension, and thus that we may solve the lifting problem (h, k) into p , since this latter map is an inner complicial fibration by assumption, to give the map m .

So we obtain the desired solution to our original lifting problem (f, g) by composing m with the inclusion $W_1 \xrightarrow{\subseteq_r} W_2$. However we know, by the comment in Definition 23 and the fact that A is a weak complicial set, that p has the RLP with respect to the elementary thinness extension $\Delta^0[2]' \xrightarrow{\subseteq_e} \Delta^0[2]''$. Combining this with the lifting property established above, it follows that p also has the RLP with respect to the thin horn inclusion $\Lambda^0[2]' \xrightarrow{\subseteq_r} \Delta^0[2]''$.

Now we know that the inclusion $i : E_1^- \hookrightarrow E_2'$, of Definition 52 is constructed by composing $E_1^- \xrightarrow{\subseteq_r} E_2^-$ with a pushout of $E_1^+ \xrightarrow{\subseteq_r} E_2^-$ and that each of these latter inclusions may be constructed as a pushout of $\Delta^0[2]' \xrightarrow{\subseteq_r} \Delta^0[2]''$ as in Observation 43. So it follows immediately, from the result of the last paragraph, that p has the RLP with respect to $i : E_1^- \hookrightarrow E_2'$ and that we may thus apply Lemma 53 to show that it also has the RLP with respect to $E_1^- \xrightarrow{\subseteq_r} E_3^-$. Finally the assumption that p also has the RLP with respect to $E_0^- \xrightarrow{\subseteq_r} E_1^-$ allows us to apply Lemma 47 and show that p is a complicial fibration as required. \square

Theorem 56. Suppose that the stratified set A is **almost** a weak inner complicial set, in the sense that we insist that it has the RLP with respect to all inner elementary anodyne extensions except $\Delta[2]' \xrightarrow{\subseteq_e} \Delta[2]''$. Furthermore, suppose that its set of thin 1-simplices is the subset

$$\{v \in A_1 \mid \exists f : \text{th}_1(\tilde{E}_2) \rightarrow A \text{ with } f(e_1^-) = v\} \quad (9)$$

of those 1-simplices with equivalence inverses then A is a weak complicial set.

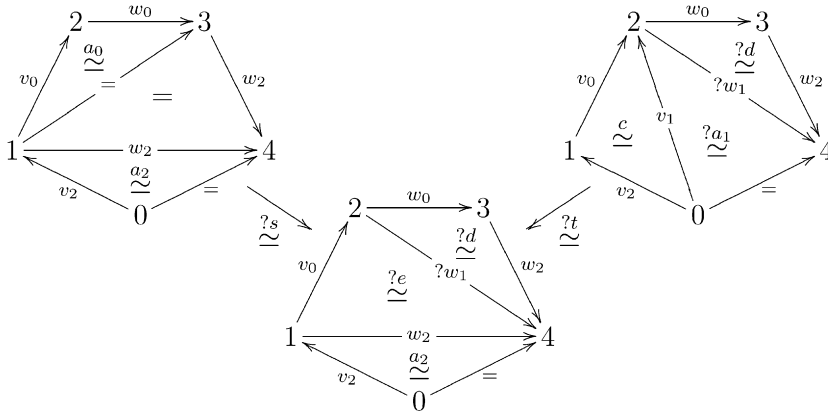
Proof. The involution $\neg : E \rightarrow E$ of Observation 44 restricts to an involution $\neg : E_2 \rightarrow E_2$ which carries e_1^- to e_1^+ . So it follows, from the description of the thin 1-simplices of A given in the statement, that if $f : \text{th}_1(\tilde{E}_2) \rightarrow A$ is a stratified map then both of $f(e_1^-)$ and $f(e_1^+) = (f \circ \neg)(e_1^-)$ are thin in A and thus that A has the RLP with respect to the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$. Consequently, we may apply Corollary 54 to show that A is a weak complicial set so long as we have verified that it is a weak inner complicial set, that is we need to show that it

also has the RLP with respect to the elementary thinness extension $\Delta^1[2]' \xrightarrow{\subseteq_e} \Delta^1[2]''$.

In other words, we must demonstrate that if $c \in A$ is a thin 2-simplex and its 1-dimensional faces $v_0 \stackrel{\text{def}}{=} c \cdot \delta_0$ and $v_2 \stackrel{\text{def}}{=} c \cdot \delta_2$ are both thin then so is $v_1 \stackrel{\text{def}}{=} c \cdot \delta_1$. However, we know that the 1-simplex v_0 (resp. v_2) is thin in A iff we have corresponding thin 2-simplices a_0, a'_0 (resp. a_2, a'_2) as depicted in display (8), so our task will be to construct similar thin simplices a_1 and a'_1 for v_1 .

We illustrate the construction of these in Fig. 2 which depicts a stratified map with domain $X \subseteq_r \text{th}_1(\Delta[4])$ the regular subset generated by the 3-simplices δ_0 and δ_3 and with codomain A . To aid our discussion 0-simplices in this diagram have been labelled to identify them in the domain X whereas all other simplices take names intended to represent simplices of the codomain A . Furthermore, a question mark appended to the front of a simplex label indicates that the corresponding simplex will be constructed in A by filling some complicial horn. We identify other simplices in X by listing their vertices so, for instance, 014 denotes the (unique) 2-simplex whose 0-dimensional faces (vertices) are 0, 1 and 4.

We commence the building of this map by initialising the left-hand pentagon with the data we are given and then working rightward, filling complicial horns as we go. So we map 123 to a_0 , 014 to a_2 and 134 to the degenerate simplex $w_2 \cdot \sigma_0$, whose faces are mutually compatible as shown in the diagram. Now in the middle pentagon we may fill the 1-complicial horn on vertices 2, 3, 4 thereby constructing a mapping of 234 to the thin 2-simplex $d \in A$ and obtaining a new 1-face $w_1 \in A$. This completes the data for a 2-complicial horn on the vertices 1, 2, 3, 4, which

Fig. 2. Composing thin 1-simplices in A .

we fill to map 1234 to a thin 3-simplex $s \in A$ and obtain a new 2-face $e \in A$ (which is thin since all the other 2-faces of s are thin). Finally, we may map 012 to the thin 2-simplex $c \in A$ that we started with in the second paragraph of this proof. In doing so we complete the data for a 1-complicial horn on the vertices 0, 1, 2, 4, which we fill to map 0124 to a thin 3-simplex $t \in A$ and obtain a new 2-face $a_1 \in A$ (which is thin since all the other 2-faces of t are thin). This is the thin 2-simplex we seek, witnessing that w_1 is a left equivalence inverse of v_1 . Dually we may replay the construction above in the alternating dual A° to derive a thin 2-simplex a'_1 which demonstrates that the 1-simplex v'_1 obtained by “composing” v'_0 and v'_2 is a left equivalence inverse of w_1 as required. \square

Example 57 (*Quasi-categories as weak complicial sets*). We are now in a position to validate Example 16 by showing that any quasi-category A may be given a stratification that makes it into a 1-trivial (cf. Notation 13, p. 1088) weak complicial set.

First note that the quasi-categorical inner horn filler conditions simply translate to postulating that the 1-trivialisation $\text{th}_1(A)$ is almost a weak inner complicial set (in the sense of the last corollary). This property places no restriction on thin 1-simplices, so we may extend the stratification of $\text{th}_1(A)$ without disrupting it by making thin all 1-simplices in the subset shown in display (9), thereby giving a stratified set we denote by A^e . It follows that we may apply Theorem 56 to this latter stratification to show that A^e is a 1-trivial weak complicial set as suggested.

Consequently, whenever we speak of quasi-categories in future we will implicitly assume that they carry the stratification of the last paragraph. The construction is clearly functorial, thereby demonstrating that we may identify the category of quasi-categories with a certain full subcategory of the category of 1-trivial weak complicial sets. Indeed Theorem 56 tells us, amongst other things, that we may characterise the objects of this full subcategory as being those 1-trivial weak complicial sets A which have the RLP with respect to the inclusion $\text{th}_1(\tilde{E}_2) \xrightarrow{\subseteq_e} E_2$.

5. Gray tensor products

In this section we generalise the complicial theory of Gray tensor products and their closures, as presented in [24], to the weak complicial context. While many of the proofs given there gen-

eralise directly we still feel that an independent presentation is warranted here, since it simplifies some aspects of the strict theory and recasts it more clearly as a piece of homotopy theory.

5.1. Some tensor products of stratified sets

Observation 58 (*A motivating analogy with bicategory theory*). In the theory of bicategories, as explicated by Street in [19], the (strict, algebraic) cartesian product of bicategories makes the category of bicategories and homomorphisms (pseudo-functors) into a symmetric monoidal category. Given a pair of bicategories $\underline{\mathcal{B}}$ and $\underline{\mathcal{C}}$ we may form the bicategory $\mathcal{H}\text{om}(\underline{\mathcal{B}}, \underline{\mathcal{C}})$ of homomorphisms, strong transformations (pseudo-naturals) and modifications between them, which provides this monoidal structure with a *weak closure* in the sense that there is a canonical biequivalence $\mathcal{H}\text{om}(\underline{\mathcal{B}} \times \underline{\mathcal{C}}, \underline{\mathcal{D}}) \simeq_b \mathcal{H}\text{om}(\underline{\mathcal{B}}, \mathcal{H}\text{om}(\underline{\mathcal{C}}, \underline{\mathcal{D}}))$. In other words, in bicategory theory the cartesian product takes the role of the Gray tensor product in 2-category theory. This insight motivates the next three definitions:

Definition 59 (*Gray tensor product of stratified sets*). The *Gray tensor product* of stratified sets X and Y is simply defined to be their cartesian product $X \otimes Y$ in the category Strat. Explicitly, $X \otimes Y$ is the stratified set whose n -simplices are pairs of n -simplices (x, y) with $x \in X_n$ and $y \in Y_n$, whose simplicial action is given pointwise $(x, y) \cdot \alpha = (x \cdot \alpha, y \cdot \alpha)$ and whose thin simplices are those (x, y) with x thin in X **and** y thin in Y .

We have two reasons for not adopting the usual cartesian product notation \times for the Gray tensor product. Firstly we would like to stress that we are primarily interested in regarding this as the appropriate generalisation of the 2-categorical Gray tensor product, the fact that it actually coincides with the categorical product in this context is an important but secondary fact. Secondly, it helps us to avoid certainly notational difficulties which might arise when manipulating simplicial sets X and Y under the minimal stratification convention, since then it is not the case that the minimal stratification of their simplicial cartesian product $X \times Y$ coincides with the stratification of their Gray tensor product $X \otimes Y$ as stratified sets. In other words, the minimal stratification operation does not preserve cartesian products.

Observation 12 reminds us that Strat is a quasi-topos, so in particular it is cartesian closed with closure (function space construction) between stratified sets X and Y denoted by $\text{hom}(X, Y)$. This is often referred to as the *stratified set of strong transformations*, since it is the true weak complicial analogue of the bicategory theorist's bicategory of homomorphisms, strong transformations and modifications. We also denote the corresponding corner product and closure by \otimes_c and hom^c respectively (cf. Recollection 120, p. 1145).

Definition 60 (*Partition operators*). We say that a pair $p, q \in \mathbb{N}$ is a *partition* of $n \in \mathbb{N}$ if $p + q = n$ and associate with it four *partition operators*:

- face operators $\perp_1^{p,q} : [p] \rightarrow [n]$ given by $\perp_1^{p,q}(i) = i$ and $\perp_2^{p,q} : [q] \rightarrow [n]$ given by $\perp_2^{p,q}(j) = j + p$, and
- degeneracy operators $\top_1^{p,q} : [n] \rightarrow [p]$ and $\top_2^{p,q} : [n] \rightarrow [q]$ given by

$$\top_1^{p,q}(i) = \begin{cases} i & \text{when } i \leq p, \\ p & \text{when } i > p \end{cases} \quad \text{and} \quad \top_2^{p,q}(i) = \begin{cases} 0 & \text{when } i < p, \\ i - p & \text{when } i \geq p \end{cases}$$

respectively.

Definition 61 (*Associative lax Gray tensor product of stratified sets*). The (associative) lax Gray tensor product $X \otimes Y$ of stratified sets X and Y (Definition 128 of [24]) is formed by taking the product of underlying simplicial sets and endowing it with the stratification under which the n -simplex (x, y) is thin in $X \otimes Y$ iff for each partition p, q of its dimension we either have that $x \cdot \perp_1^{p,q}$ is a thin p -simplex in X or that $y \cdot \perp_2^{p,q}$ is a thin q -simplex in Y .

Notice, in particular, that by definition stratifications can have no thin 0-simplices so this condition applied to the extremal partitions $n, 0$ and $0, n$ imply that if (x, y) is thin in $X \otimes Y$ then x is thin in X and y is thin in Y . In other words, $X \otimes Y$ is an entire subset (cf. Definition 9, p. 1087) of $X \otimes Y$.

Observation 62 (*Lax Gray tensors in strict complicial set theory*). In the theory of (strict) complicial sets presented in [24], the relationship between the lax Gray tensors of stratified sets and (strict) ω -categories extends well beyond mere analogy. Indeed, Section 11.4 of that work demonstrates that the lax Gray tensor product of (strict) ω -categories, as defined by Steiner [18] or Crans [5], may be obtained by reflecting the lax Gray tensor of stratified sets to the equivalent subcategory of (strict) complicial sets.

Observation 63. The primary properties of \otimes as a tensor product on Strat are established in Lemmas 129 and 131 of [24] and may be derived directly from the *partition identities* between partition operators given in Notation 5 of that work. In summary, \otimes may be extended to stratified maps and equipped with canonical associativity and identity isomorphisms which make it into the tensor of a (non-symmetric) monoidal structure on Strat. This structure is completely characterised by the fact that the forgetful underlying simplicial set functor becomes a strict monoidal functor from the monoidal category (Strat, \otimes , $\Delta[0]$) to the cartesian closed category (Simp, \times , $\Delta[0]$). Furthermore, \otimes is well behaved with respect to alternating duals (cf. Observation 26, p. 1093), with the “swap” map on underlying simplicial products providing us with canonical isomorphisms $(X \otimes Y)^\circ \cong Y^\circ \otimes X^\circ$.

However, as discussed in Observation 136 of [24], while \otimes provides Strat with a genuine monoidal structure it fails to be well behaved with respect to colimits of stratified sets. This leads us to define the following, closely related, tensor product for which left and right tensoring does preserve colimit but which fails to be coherently associative. To simplify our presentation here a little the next definition does not quite follow that of the corresponding construction of that work, but nevertheless shares all of its important properties.

Definition 64 (*Lax Gray pre-tensor product of stratified sets*). The lax Gray pre-tensor product $X \boxtimes Y$ of stratified sets X and Y (Definition 135 of [24]) is formed by taking the product of underlying simplicial sets and endowing it with a stratification under which an r -simplex $(x, y) \in X \times Y$ is thin if either

- there exists $0 < k < r$ such that $x = x' \cdot \sigma_{k-1}$ and $y = y' \cdot \sigma_k$ for some pair of simplices $x' \in X$ and $y' \in Y$, or
- there exists a partition p, q of its dimension and simplices $x' \in X_p$ and $y' \in Y_q$ such that $x = x' \cdot \perp_1^{i,j}$ and $y = y' \cdot \perp_2^{i,j}$ and either x' is thin in X or y' is thin in Y .

It is easily demonstrated that this is a stratification which makes $X \boxtimes Y$ into an entire subset of $X \otimes Y$.

Observation 65. Here again, it is easily shown that the action of cartesian product as a bifunctor of underlying simplicial sets may be lifted to make \boxtimes into a bifunctor on \mathbf{Strat} . This time, however, it is **not** the case that canonical associativity isomorphisms also lift in this way, but it is still true that identity isomorphisms lift to give $X \boxtimes \Delta[0] \cong X \cong \Delta[0] \boxtimes X$ and that the “swap” map provides a canonical isomorphism $(X \boxtimes Y)^\circ \cong Y^\circ \boxtimes X^\circ$.

Most importantly, Lemma 142 of [24] demonstrates that the pre-tensor \boxtimes preserves colimits in each variable. Consequently, since \mathbf{Strat} is locally finitely presentable, it follows that it possesses closures $\text{lax}_l(X, Z)$ and $\text{lax}_r(Y, Z)$ which are right adjoint to the endo-functors $X \boxtimes -$ and $- \boxtimes Y$ respectively. We often call these the *stratified set of left (resp. right) lax transformations* since they generalise the bicategory theorist’s bicategories of homomorphisms, left (resp. right) lax transformations and modifications. Again we adopt the notations \boxtimes_c , lax_l^c and lax_r^c to denote the corresponding corner tensor and its closures (cf. Recollection 120, p. 1145).

Observation 66. Finally, it is also worth pointing out that Lemma 139 of [24] demonstrates that for each pair of stratified sets X and Y the entire inclusion $X \boxtimes Y \xrightarrow{\subseteq_e} X \otimes Y$ is an inner anodyne extension. In other words, we might say that weak complicial sets “do not see” the difference between the tensors \boxtimes and \otimes .

We may also define a bifunctor $\boxdot : \mathbf{Strat} \times \mathbf{Strat} \rightarrow \mathbf{Strat}$ for which $X \boxdot Y$ is the entire superset of $X \boxtimes Y$ constructed by making thin all simplices of the form $(x \cdot \prod_1^{r,s}, y \cdot \prod_2^{r,s})$ with $x \in X$, $y \in Y$ and $r, s > 0$. Notice that each of these simplices is thin in $X \otimes Y$, so we have an entire inclusion $X \boxdot Y \xrightarrow{\subseteq_e} X \otimes Y$. Furthermore, as the reader may readily verify, we may modify the proof given in Lemma 139 of [24] to show that this inclusion is also an inner anodyne extension (cf. Definition 22, p. 1091).

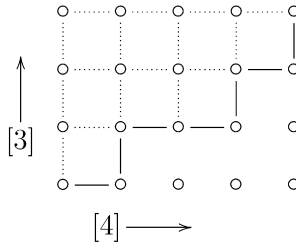
5.2. Gray tensors and anodyne extensions

Our primary interest in the remainder of this section will be to demonstrate that these tensors are well behaved with respect to certain anodyne extensions. In the process we show that the functors $\text{hom}(X, -)$, $\text{lax}_l(X, -)$ and $\text{lax}_r(Y, -)$ all preserve weak (inner) compliciality.

Definition 67. As ever, the non-degenerate $(n + m)$ -simplices of the simplicial set $\Delta[n] \times \Delta[m]$ are called *shuffles*. An easy and useful characterisation of these is that they are precisely the $(n + m)$ -simplices (α, β) which satisfy the *ordinate summation property* which states that $\alpha(i) + \beta(i) = i$ for all $i \in [n + m]$. We define the *depth* (cf. Porter and Ehlers [9]) of such a shuffle to be the integer:

$$\text{dp}(\alpha, \beta) \stackrel{\text{def}}{=} \sum_{i=0}^{n+m} \min(\alpha(i), m - \beta(i)).$$

Observation 68 (More about shuffles). We may depict a shuffle in $\Delta[n] \times \Delta[m]$ as a path of strictly horizontal (rightward) and vertical (upward) moves on an $[n] \times [m]$ grid, which starts at its bottom left corner and ends at its top right one. Then, as observed in [9], the depth of that shuffle is simply the number of squares of that grid which occur to the left of and above that path. For instance the depth of the example (solid line) in Fig. 3 is equal to the number of squares that have been outlined with dotted boundaries, which in this case is 8.

Fig. 3. A shuffle in $\Delta[4] \times \Delta[3]$.

In line with this depiction, we make the following simple observations and definitions:

- (a) The only depth 0 shuffle is the one which would be depicted as a sequence of m upward moves followed by n rightward ones, in other words the simplex $(\Pi_2^{m,n}, \Pi_1^{m,n})$.
- (b) All shuffles have depth less than or equal to nm and the only depth nm shuffle is the one which would be depicted as a sequence of n rightward moves followed by m upward ones, in other words the simplex $(\Pi_1^{n,m}, \Pi_2^{n,m})$.
- (c) If (α, β) is a shuffle we say that its i th vertex (for $0 < i < n + m$) is a *left-upper corner* if $\alpha(i - 1) = \alpha(i)$ and $\beta(i) = \beta(i + 1)$ and we say it is a *right-lower corner* if $\beta(i - 1) = \beta(i)$ and $\alpha(i) = \alpha(i + 1)$. These are simply the right and left-handed right angle turning points in its depiction.
- (d) The only shuffle with no left-upper corners is the maximal depth shuffle $(\Pi_1^{n,m}, \Pi_2^{n,m})$. Dually the only shuffle with no right-lower corners is the minimal depth shuffle $(\Pi_2^{m,n}, \Pi_1^{m,n})$.
- (e) No two left-upper (resp. right-lower) corners of (α, β) can be immediately adjacent, that is to say if $i < j$ are indices of two such left upper corners then we actually have $i + 1 < j$.
- (f) The i th vertex of our shuffle is neither a left-upper nor a right-lower corner iff the face $(\alpha, \beta) \cdot \delta_i$ is a simplex of $\partial(\Delta[n] \times \Delta[m])$.

Notation 69. In the next lemma we will assume that P is a stratified set with underlying simplicial set $\Delta[n] \times \Delta[m]$ and which satisfies the condition that whenever (ϕ, ψ) is a non-degenerate r -simplex of P and l is some integer with $0 < l < r$ such that $\phi(l - 1) = \phi(l)$ and $\psi(l) = \psi(l + 1)$ (upper-left corner) then:

- (a) (ϕ, ψ) is thin in P , and
- (b) if the face $(\phi, \psi) \cdot \delta_l$ is thin in P then so are $(\phi, \psi) \cdot \delta_{l-1}$ and $(\phi, \psi) \cdot \delta_{l+1}$.

The reader might like to compare these conditions to the corresponding clauses of Definition 27, as we do in detail in the proof of Lemma 70 later on.

Most importantly, if we are given entire supersets N and M of $\Delta[n]$ and $\Delta[m]$ respectively then each of the stratified sets $N \otimes M$ and $N \circledast M$ satisfies the condition required of P in the last paragraph. Indeed, it is also the case that $\Delta[n] \otimes \Delta[m]$ is minimal for those conditions, in the sense that it is an entire subset of any P which satisfies them. The proofs of these facts are a matter of routine combinatorial verification, directly from the definitions of \circledast and \otimes , which we leave to the reader. As a guide, Sections 7 and 8 of [24] contain numerous examples of detailed calculations involving the stratification of the lax Gray tensor product.

We will also have reason to consider the following stratified subsets of P :

- ∂P , the *boundary* of P , which is the regular subset whose underlying simplicial set is $\partial(\Delta[n] \times \Delta[m]) \stackrel{\text{def}}{=} (\partial\Delta[n] \times \Delta[m]) \cup (\Delta[n] \times \partial\Delta[m]) \subseteq_s \Delta[m] \times \Delta[n]$.
- P_d which is the regular subset generated by the set of shuffles in P of depth less than or equal to $d \in [n + m]$.
- ∂P_d , the boundary of P_d , which is the intersection of P_d and ∂P .
- \tilde{P}_d and $\partial \tilde{P}_d$ which are the entire subsets of P defined by $\tilde{P}_d \stackrel{\text{def}}{=} (\Delta[n] \otimes \Delta[m]) \cup P_d$ and $\partial \tilde{P}_d \stackrel{\text{def}}{=} (\Delta[n] \otimes \Delta[m]) \cup \partial P_d$.

Before moving on, it is worth noting that by convention we take P_{-1} to be empty and that P_{nm} is equal to P itself. Furthermore, it is easily seen that P_{nm-1} is precisely the regular subset of P containing those simplices which do not have $(n, 0)$ as a vertex. This latter set, its boundary and their associated unions with $\Delta[n] \otimes \Delta[m]$ are our real objects of interest in the following lemma (and its corollary) and so we adopt the denotations P_\bullet , ∂P_\bullet , \tilde{P}_\bullet and $\partial \tilde{P}_\bullet$ for these in order to avoid tedious repetition of the index $nm - 1$.

Lemma 70. *For each integer $d \in \mathbb{N}$ with $0 \leq d < nm$ the regular inclusion $P_{d-1} \cup \partial P_d \xrightarrow{\subseteq_r} P_d$ (cf. Notation 69, p. 1117) is an inner anodyne extension. It follows that the regular inclusion $\partial P_\bullet \xrightarrow{\subseteq_r} P_\bullet$ is also an inner anodyne extension.*

Proof. This result depends on a few simple combinatorial observations:

Observation (i). *If (α, β) is a shuffle in P and $0 < t < n + m$ is an integer with $\beta(t - 1) = \beta(t)$ and $\alpha(t) = \alpha(t + 1)$ (right-lower corner) then $(\alpha, \beta) \cdot \delta_t$ is a face of some shuffle of lower depth.*

Observe that the ordinate summation property of shuffles given in Definition 67 may be applied to the conditions on α and β at t in the statement to establish that $\alpha(t) = \alpha(t - 1) + 1$ and $\beta(t + 1) = \beta(t) + 1$. It follows easily that we may construct a well defined $(n + m)$ -simplex (α', β') by letting

$$\alpha'(i) = \begin{cases} \alpha(i) & \text{if } i \neq t, \\ \alpha(t) - 1 & \text{if } i = t \end{cases} \quad \text{and} \quad \beta'(i) = \begin{cases} \beta(i) & \text{if } i \neq t, \\ \beta(t) + 1 & \text{if } i = t \end{cases}$$

which simply turns the original right-lower corner in (α, β) into a left-upper corner in (α', β') . Now it is clear that this again satisfies the ordinate summation property, making it a shuffle which only differs from our original one at t and thus has $(\alpha', \beta') \cdot \delta_t = (\alpha, \beta) \cdot \delta_t$. Furthermore, the expressions for the depths of these only differ at t where $\min(\alpha'(t), m - \beta'(t)) = \min(\alpha(t), m - \beta(t)) - 1$ so it follows that $\text{dp}(\alpha', \beta') = \text{dp}(\alpha, \beta) - 1$ as required.

Observation (ii). *Suppose that (α, β) and (α', β') are distinct shuffles of the same depth d say then any face common to both of them is an element of P_{d-1} .*

Observe that the integer

$$s \stackrel{\text{def}}{=} \max\{i \in [n + m] \mid (\forall j \leq i) \alpha(j) = \alpha'(j)\}$$

is well defined, since all shuffles have $(0, 0)$ as their 0th vertex, and that it is less than $n + m$, because our shuffles are distinct. Notice also that the ordinal summation property combined with the definition of s ensures that our shuffles agree at all vertices up to and including their s th ones and that we may assume without loss of generality that $\alpha(s + 1) = \alpha(s) + 1$, $\beta(s + 1) = \beta(s)$ and $\alpha'(s + 1) = \alpha'(s) = \alpha(s)$, $\beta'(s + 1) = \beta'(s) + 1 = \beta(s) + 1$, by swapping the identities of our shuffles if necessary. Now we also know that the integer

$$t \stackrel{\text{def}}{=} \max\{i \in [n + m] \mid i > s \wedge \beta(i) = \beta(s)\}$$

is well defined, since the set over which we are taking this maximum contains $s + 1$, and it must be less than $n + m$, because $\beta(s + 1) < \beta'(s + 1) \leq m$ and $m = \beta(n + m)$ by the ordinate summation property. By construction our shuffles disagree at t , since $\beta(t) = \beta(s) = \beta'(s) < \beta'(s + 1) \leq \beta'(t)$, so any face common to both of them must be a face of $(\alpha, \beta) \cdot \delta_t$. Furthermore the maximality of t combined with the ordinal summation property implies that we also have $\beta(t - 1) = \beta(t)$ and $\alpha(t) = \alpha(t + 1)$ (right-lower corner), so we may apply the previous observation to show that the simplex $(\alpha, \beta) \cdot \delta_t$ is a face of a shuffle of depth less than $d = \text{dp}(\alpha, \beta)$ and that it is thus an element of P_{d-1} . Now, the shuffles (α, β) and (α', β') disagree at their t th vertex, so any face (ϕ, ψ) common to both of them must also be a face of $(\alpha, \beta) \cdot \delta_t \in P_{d-1}$ and must therefore, in turn, be an element of the stratified set P_{d-1} as required.

Observation (iii). *If (α, β) is a shuffle of depth d then we may factor the corresponding Yoneda map $\ulcorner(\alpha, \beta)\urcorner: \Delta[n + m] \rightarrow P_d$ as a composite of an entire inclusion $\Delta[n + m] \xrightarrow{\subseteq_e} N$ and a regular inclusion $\ulcorner(\alpha, \beta)\urcorner: N \hookrightarrow P_d$. The entire superset N of $\Delta[n + m]$ is a \vec{k} -complicial $(n + m)$ -simplex (see Definition 27, p. 1094) with respect to the family*

$$\vec{k} = \{k \in \mathbb{N} \mid 0 < k < n + m \wedge \alpha(k - 1) = \alpha(k) \wedge \beta(k) = \beta(k + 1)\}$$

of (indices of) the left-upper corners of (α, β) .

Firstly it is clear that the simplicial set $\Delta[n] \times \Delta[m]$ enjoys the property that the Yoneda maps associated with its non-degenerate simplices are all simplicial inclusions. So applying the entire coimage factorisation of Definition 9 to the Yoneda map associated with the shuffle (α, β) of P_d we obtain the entire superset N of $\Delta[n + m]$ and the regular inclusion $N \hookrightarrow P_d$ of the statement. In other words, a simplex γ is thin in N if and only if its image $(\alpha, \beta) \cdot \gamma = (\alpha \circ \gamma, \beta \circ \gamma)$ under the Yoneda map $\ulcorner(\alpha, \beta)\urcorner$ is thin in P_d . Now index the elements of \vec{k} in increasing order $\{k_1 < k_2 < \dots < k_t\}$ and notice that Observation 68 ensures that this satisfies the conditions of Definition 27, in particular its point (d) tells us that \vec{k} is non-empty, because by assumption the depth d of our shuffle is less than mn , whereas its point (e) implies that for each index $1 \leq i < t$ we have $k_i + 1 < k_{i+1}$. Notice also that all of the elements of \vec{k} are greater than 0 and less than n , so all of our arguments here will involve inner (generalised) horns.

Next let us examine the stratification of N in more detail. First, suppose γ is a k -admissible r -simplex of $\Delta[n + m]$ (cf. Notation 10, p. 1087) for some $k \in \vec{k}$ and let $l \in [r]$ be the unique integer with $\gamma(l) = k$, $\gamma(l - 1) = k - 1$ and $\gamma(l + 1) = k + 1$ (cf. Observation 11, p. 1088). To see if γ is thin in N we consider the associated simplex $(\alpha, \beta) \cdot \gamma = (\alpha \circ \gamma, \beta \circ \gamma)$ of P_d , for which the defining property of the elements $k \in \vec{k}$ and the definition of l provides us with the equalities $\alpha \circ \gamma(l - 1) = \alpha(k - 1) = \alpha(k) = \alpha \circ \gamma(l)$ and $\beta \circ \gamma(l) = \beta(k) = \beta(k + 1) = \beta \circ \gamma(l + 1)$.

These equalities demonstrate that $(\alpha, \beta) \cdot \gamma$ is thin in P , by condition (a) of Notation 69, and thus it is thin in the regular subset $P_d \subseteq_r P$. Consequently γ is thin in N and thus, quantifying over $k \in \vec{k}$ and all k -admissible simplices $\gamma \in \Delta[n + m]$, we have demonstrated that N satisfies condition (a) of Definition 27. However, we can take this argument a bit further and observe that if $\gamma \circ \delta_l$ is thin in N then $(\alpha, \beta) \cdot (\gamma \circ \delta_l) = (\alpha \circ \gamma, \beta \circ \gamma) \cdot \delta_l$ is thin in $P_d \subseteq_r P$, so we may apply the condition (b) of Notation 69 to show that $(\alpha, \beta) \cdot (\gamma \circ \delta_{l-1}) = (\alpha \circ \gamma, \beta \circ \gamma) \cdot \delta_{l-1}$ and $(\alpha, \beta) \cdot (\gamma \circ \delta_{l+1}) = (\alpha \circ \gamma, \beta \circ \gamma) \cdot \delta_{l+1}$ are also both thin in P_d and thus that $\gamma \circ \delta_{l-1}$ and $\gamma \circ \delta_{l+1}$ are thin in N as required by condition (b) of Definition 27. In other words, we have shown that N is a \vec{k} -complicial $(n + m)$ -simplex as required.

Observation (iv). *The generalised horn $\Lambda^{\vec{k}}N \subseteq_r N$ (cf. Definition 27, p. 1094) is the inverse image of the regular subset $P_{d-1} \cup \partial P_d \subseteq_r P_d$ along the inclusion $\ulcorner(\alpha, \beta)\urcorner: N \hookrightarrow P_d$.*

The inverse image of a regular subset along any stratified map is always a regular subset, so all we need do is check that the inverse image $L \subseteq_r N$ of $P_{d-1} \cup \partial P_d \subseteq_r P_d$ along $\ulcorner(\alpha, \beta)\urcorner: H \hookrightarrow P_d$ contains the same simplices as $\Lambda^{\vec{k}}N$. So to show that $\Lambda^{\vec{k}}N \subseteq_r L$ we recall that $\Lambda^{\vec{k}}N$ is the regular subset generated by the set of faces $\{\delta_i \mid i \in [n] \setminus \vec{k}\}$ and infer that it is enough to show that each of these is a simplex of L . However, consulting point (f) of Observation 68 we see that if an integer $i \in [n]$ is not in \vec{k} , that is to say it is not the index of a left-upper corner, then it must either be the index of a right-lower corner, in which case we may apply Observation (i) of this proof to show that $(\alpha, \beta) \cdot \delta_i$ is in P_{d-1} , or it must be an element of the boundary ∂P_d . In other words, for each $i \in [n] \setminus \vec{k}$ the face $(\alpha, \beta) \cdot \delta_i$ is in $P_{d-1} \cup \partial P_d$ and so δ_i is in its inverse image L as required.

Conversely, we know that a simplex $\gamma \in N$ is not an element of $\Lambda^{\vec{k}}N$ if and only if each element of $[n] \setminus \vec{k}$ is also an element of $\text{im}(\gamma)$, or equivalently iff any element of $[n]$ which is not in $\text{im}(\gamma)$ is in \vec{k} . It then follows, from the definition of \vec{k} with respect to (α, β) , that each of the operators $\alpha \circ \gamma$ and $\beta \circ \gamma$ are surjective and in particular we know that $\ulcorner(\alpha, \beta)\urcorner(\gamma) = (\alpha \circ \gamma, \beta \circ \gamma)$ is not an element of the boundary ∂P_d . Furthermore, suppose that (α', β') is any other shuffle that has $(\alpha \circ \gamma, \beta \circ \gamma)$ as a face and observe that we then have $\alpha \circ \gamma = \alpha' \circ \gamma$ and $\beta \circ \gamma = \beta' \circ \gamma$, so if we consider an index k at which (α, β) and (α', β') differ, it follows that it cannot be an element of $\text{im}(\gamma)$ and thus that it must be an element of \vec{k} . Conversely, we know that if k is in \vec{k} then neither $k - 1$ nor $k + 1$ can be in there so it follows that they must both be in $\text{im}(\gamma)$ and thus that (α, β) and (α', β') must agree at those indices. Now since k is in \vec{k} we know that (α, β) has a left-upper corner there and since our two shuffles agree at $k - 1$ and $k + 1$ but disagree at k it is clear that (α', β') must have a right-lower corner at that index. It follows, therefore, that $\alpha'(k) = \alpha(k) + 1$ and $\beta'(k) = \beta(k) - 1$ and thus that $\min(\alpha'(k), m - \beta'(k)) = \min(\alpha(k), m - \beta(k)) + 1$, so since this is true at any index where these shuffles differ it follows that $d = \text{dp}(\alpha, \beta) < \text{dp}(\alpha', \beta')$. This demonstrates that $\ulcorner(\alpha, \beta)\urcorner(\gamma) = (\alpha \circ \gamma, \beta \circ \gamma)$ is not an element of P_{d-1} , since we have shown that any shuffle of which it is a face must have depth at least d , so combining this with the corresponding fact with respect to the boundary ∂P_d we find that γ cannot be an element of the inverse image L of $P_{d-1} \cup \partial P_d$ along $\ulcorner(\alpha, \beta)\urcorner$ as required.

Now we may apply these observations to proving the result described in the first sentence of the statement. To do so enumerate the shuffles of depth d over a suitable index set $I = \{1, 2, \dots, s\}$ and for each $i \in I$ let N_i denote the entire superset of $\Delta[n + m]$ and \vec{k}_i denote the family of integers associated with the i th shuffle (α_i, β_i) as in Observation (iii). Now define

an increasing sequence of regular subsets X_i of P_d by starting at $X_0 = P_{d-1} \cup \partial P_d$ and letting each successive X_i be the smallest regular subset of P_d containing its predecessor X_{i-1} and the shuffle (α_i, β_i) , thus ensuring that the last member of this sequence X_s is actually equal to P_s itself. Now suppose that γ is an r -simplex in N_i and consider the face $(\alpha_i, \beta_i) \cdot \gamma \in P_d$ which is its image under the regular inclusion $\lceil (\alpha_i, \beta_i) \rceil: N_i \hookrightarrow P_d$ of Observation (iii). If this is an element of X_{i-1} then, by definition, it must either be an element of $P_{d-1} \cup \partial P_d$ or it must also be a face of some other shuffle (α_j, β_j) with $j \leq i-1$, and in the latter case we may apply Observation (ii) to show that it is again an element of P_{d-1} . Consequently, the inverse image of $X_{i-1} \subseteq_r P_d$ along $\lceil (\alpha_i, \beta_i) \rceil: N_i \hookrightarrow P_d$ coincides with the inverse image its regular subset $P_{d-1} \cup \partial P_d$ along the same map which we know, by Observation (iv), is the generalised horn $\Lambda^{\bar{k}_i} N_i \subseteq_r N_i$. Furthermore, the definition of X_i may be trivially recast to say that it is the union of X_{i-1} and the direct image of N_i under the regular inclusion $\lceil (\alpha_i, \beta_i) \rceil: N_i \hookrightarrow P_d$. Summarising these facts by applying Recollection 24 we obtain a glueing square

$$\begin{array}{ccc} \Lambda^{\bar{k}_i} N_i & \xrightarrow{\subseteq_r} & N_i \\ \downarrow \lceil & \lrcorner & \downarrow \lceil (\alpha_i, \beta_i) \rceil \\ X_{i-1} & \xrightarrow{\subseteq_r} & X_i \end{array}$$

which demonstrates that its lower horizontal inclusion is an inner anodyne extension since its upper horizontal is the inner anodyne extension of Lemma 28. It follows, therefore, that $P_{d-1} \cup \partial P_d \xrightarrow{\subseteq_r} P_d$ may be decomposed as a composite of inner anodyne extensions $X_{i-1} \xrightarrow{\subseteq_r} X_i$ and is thus itself an inner anodyne extension as required.

Finally, to prove the last sentence of the statement observe that we have equalities $P_d \cup (P_{d-1} \cup \partial P_\bullet) = P_d \cup \partial P_\bullet$ and $P_d \cap (P_{d-1} \cup \partial P_\bullet) = P_{d-1} \cup \partial P_d$ so we may apply Recollection 24 to obtain a glueing square which displays the regular inclusion $P_{d-1} \cup \partial P_\bullet \xrightarrow{\subseteq_r} P_d \cup \partial P_\bullet$ as a pushout of the anodyne extension $P_{d-1} \cup \partial P_d \xrightarrow{\subseteq_r} P_d$. It follows therefore that we have a sequence of regular subsets $P_d \cup \partial P_\bullet$ of P_\bullet ($d = -1, 0, 1, \dots, nm-1$) whose first member is ∂P_\bullet , whose last is P_\bullet and in which each inclusion of a sequence member into its successor was shown to be an anodyne extension (as a pushout of such) in the last sentence. It follows therefore that their composite $\partial P_\bullet \xrightarrow{\subseteq_r} P_\bullet$ is also an anodyne extension as required. \square

Corollary 71. *For each integer $d \in \mathbb{N}$ with $0 \leq d < nm$ the entire inclusion $\tilde{P}_{d-1} \cup \partial P_d \xrightarrow{\subseteq_e} \tilde{P}_d$ (cf. Notation 69, p. 1117) is an inner anodyne extension. It follows that the entire inclusion $\partial \tilde{P}_\bullet \xrightarrow{\subseteq_e} \tilde{P}_\bullet$ is also an inner anodyne extension.*

Proof. This is a routine reprise of the method used in the proof of the last lemma. We simply replace pushouts of generalised horn extensions by pushouts of the generalised thinness extensions discussed in Corollary 29 wherever necessary. We leave the details to the reader. \square

In the next lemma we use the notation \odot and \odot_c to represent either one of the tensors \otimes or \otimes on Strat and its associated corner tensor.

Lemma 72. *If k is an integer with $0 \leq k < n$ then each of the corner tensors*

$$\begin{aligned} & (\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]) \odot_c (\partial \Delta[m] \xrightarrow{\subseteq_r} \Delta[m]), \\ & (\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]) \odot_c (\Delta[m] \xrightarrow{\subseteq_e} \Delta[m]_i), \\ & (\Delta^k[n]' \xrightarrow{\subseteq_r} \Delta^k[n]'') \odot_c (\partial \Delta[m] \xrightarrow{\subseteq_r} \Delta[m]), \\ & (\Delta^k[n]' \xrightarrow{\subseteq_r} \Delta^k[n]'') \odot_c (\Delta[m] \xrightarrow{\subseteq_e} \Delta[m]_i) \end{aligned}$$

is a left anodyne extension and is an inner anodyne extension if $0 < k$.

Proof. We prove the stated result for the first corner tensor in the list above in detail. Firstly arguing just as we did in Observation 37 we see that the corner tensor of the two maps in the statement is (isomorphic to) the regular subset inclusion $(\Delta^k[n] \odot \Delta[m]) \cup (\Delta^k[n] \odot \partial \Delta[m]) \xrightarrow{\subseteq_r} \Delta^k[n] \odot \Delta[m]$ and we adopt the letter R to denote the codomain of this inclusion. To prove that this is a left anodyne extension we will apply Lemma 70 *twice* to the stratified sets $Q \stackrel{\text{def}}{=} \Delta[n-1] \odot \Delta[m]$ and $P \stackrel{\text{def}}{=} \Delta^k[n] \odot \Delta[m]$ respectively. So consider the increasing sequence $R \subseteq_r R \cup \partial P_\bullet \subseteq_r R \cup P_\bullet \subseteq_r P$ of regular subset inclusions, which are subject to the following observations:

Observation (i). A simplex (α, β) is in R iff either $\text{im}(\alpha) \cup \{k\} \neq [n]$ or $\text{im}(\beta) \neq [m]$ whereas it is in ∂P_\bullet iff it does not have $(n, 0)$ as a simplex and either $\text{im}(\alpha) \neq [n]$ or $\text{im}(\beta) \neq [m]$. So under the assumption that $k < n$ we define W to be the regular subset of P of those simplices (α, β) which do not have $(n, 0)$ as a simplex and for which $k \notin \text{im}(\alpha)$ and may then easily demonstrate that $R \cup \partial P_\bullet = R \cup W$.

Observation (ii). The stratified map $\lceil \delta_k \rceil : \Delta[n-1] \hookrightarrow \Delta^k[n]$, which corresponds to the $(n-1)$ -simplex δ_k in $\Delta^k[n]$ by Yoneda's lemma, is a regular inclusion. Furthermore, each of the tensors \otimes and \boxtimes preserves inclusions and regularity, as the reader may readily verify, so it follows that we obtain a regular inclusion $\lceil \delta_k \rceil \odot \Delta^k[n]$ from $Q = \Delta[n-1] \odot \Delta[m]$ to P . Under the assumption that $k < n$ it is then easily seen that the regular subset W of the last observation is simply the direct image of $Q_\bullet \subseteq_r Q$ under this inclusion and that ∂Q_\bullet is the inverse image of $R \subseteq_r P$ along the same inclusion. It follows, by applying Recollection 24, that we have a glueing square

$$\begin{array}{ccc} \partial Q_\bullet & \xrightarrow{\subseteq_r} & Q_\bullet \\ \downarrow & \lrcorner & \downarrow \\ R & \xrightarrow[\subseteq_r]{} & R \cup W = R \cup \partial P_\bullet \end{array}$$

whose upper horizontal is an anodyne extension by Lemma 70. Consequently its pushout, the inclusion of R into $R \cup \partial P$, is also an inner anodyne extension.

Observation (iii). Clearly we have $(R \cup \partial P_\bullet) \cup P_\bullet = R \cup P_\bullet$ and $(R \cup \partial P_\bullet) \cap P_\bullet = R \cup \partial P_\bullet$, where the latter equality holds because R is a subset of the boundary ∂P , so we may apply Recollection 24 to obtain a glueing square:

$$\begin{array}{ccc} \partial P_\bullet & \xrightarrow{\subseteq_r} & P_\bullet \\ \subseteq_r \downarrow \lrcorner & & \lrcorner \downarrow \subseteq_r \\ R \cup \partial P_\bullet & \xrightarrow[\subseteq_r]{} & R \cup P_\bullet \end{array}$$

Again we may apply Lemma 70 to show that the upper horizontal here and its pushout, the inclusion $R \cup \partial P_\bullet \xrightarrow{\subseteq_r} R \cup P_\bullet$, are both inner anodyne extensions.

Observation (iv). Only two simplices of P are not elements of $R \cup P_\bullet$, those being the maximal depth shuffle $(\Pi_1^{n,m}, \Pi_2^{n,m})$ and its k th $(n+m-1)$ -dimensional face $(\Pi_1^{n,m}, \Pi_2^{n,m}) \cdot \delta_k$. Consequently we are led to considering the Yoneda map corresponding to this shuffle, which is an inclusion $\ulcorner (\Pi_1^{n,m}, \Pi_2^{n,m}) \urcorner: \Delta[n+m] \hookrightarrow P$. In fact this may be lifted to a stratified map whose domain is $\Delta^k[n+m]$, although the combinatorial details of the argument demonstrating that fact (which we leave to the reader) depend upon precisely which of the tensors \circledast or \circledotimes we are studying. Indeed these cases diverge a little further at this point, since it turns out that the face $(\Pi_1^{n,m}, \Pi_2^{n,m}) \cdot \delta_k$ is thin in P when it is defined using \circledast but that this simplex is not thin in there when we consider \circledotimes . In that first case, it turns out that the flanking faces $(\Pi_1^{n,m}, \Pi_2^{n,m}) \cdot \delta_i$ ($i \in \{k-1, k+1\} \cap [n+m]$) are also thin in P so we may lift our map further to one with domain $\Delta^k[n+m]''$. Ultimately however, regardless of tensor, we may apply Recollection 24 and obtain one of the following glueing squares:

$$\begin{array}{ccc} \Delta^k[n+m]' \xrightarrow{\subseteq_r} \Delta^k[n+m]'' & & \Delta^k[n+m] \xrightarrow{\subseteq_r} \Delta^k[n+m] \\ \downarrow \lrcorner & \lrcorner \downarrow \ulcorner (\Pi_1^{n,m}, \Pi_2^{n,m}) \urcorner & \downarrow \lrcorner \\ R \cup P_\bullet \xrightarrow[\subseteq_r]{} \Delta^k[n] \circledast \Delta[m] & & R \cup P_\bullet \xrightarrow[\subseteq_r]{} \Delta^k[n] \circledotimes \Delta[m] \end{array}$$

Consequently the inclusion $R \cup P_\bullet \xrightarrow{\subseteq_r} P$ is a pushout of a left outer (and possibly thin) horn if $k=0$ and of an inner (and possibly thin) horn otherwise and is thus a left or inner anodyne extension.

Summarising these observations, we see that each of the first two inclusions in our sequence above are inner anodyne extensions whereas the last one is a left anodyne extension if $k=0$ and an inner anodyne extension if $0 < k < n$. It follows therefore that their composite, the corner join under study, is an inner or left anodyne extension as described in the statement.

The remaining three corner tensors of the statement, each of which is an entire inclusion, may all be shown to be left or inner anodyne extensions using a routine reprise of the argument above. The primary modification required is that we replace pushouts of the inner anodyne extension of Lemma 70 by pushouts of the corresponding one of Corollary 71. We leave the details to the reader. \square

Observation 73. Notice that the assumption $k < n$ was vital to all of the observations made in the proof above. A completely different combinatorial argument would be required to directly prove the corresponding result for right outer horns. This however need not bother us here, since everything we need with regard to right compciality may be derived from Lemma 49 as we do in Theorem 75 below.

Observation 74. The Gray tensor product \otimes preserves colimits in each variable, so we may apply Observation 121 to the result of the last lemma and show that if $e : U \hookrightarrow V$ is a left (resp. inner) anodyne extension and $i : X \hookrightarrow Y$ is any inclusion then their corner join $e \otimes_c i$ is also a left (resp. inner) anodyne extension.

Things are, however, somewhat less straightforward for the lax Gray tensor product \otimes which is not well behaved with respect to colimits. Unfortunately it is also not possible to replace this by the related “colimit friendly” pre-tensor \boxtimes because this does not satisfy the conditions required to make the arguments of the last few lemmas work. We will return to resolve this issue in Section 6, for now however we have the following useful theorem for the closed structure associated with \otimes :

Theorem 75. *If A is a weak compcial set and X is any stratified set then the stratified set of strong transformations $\text{hom}(X, A)$ is also a weak compcial set. Furthermore if $p : A \twoheadrightarrow B$ is a compcial fibration between compcial sets and $X \hookrightarrow Y$ is any inclusion then the corner closure $\text{hom}^c(i, p)$ is also a compcial fibration.*

Proof. Applying Observation 122 to the result of the last observation we find that $\text{hom}(X, A)$ is a weak left compcial set to which we may apply Corollary 49 to demonstrate that it is actually a weak compcial set. A similar argument shows that the corner closure $\text{hom}^c(i, p)$ is a left compcial fibration (cf. Definition 22, p. 1091), whose codomain is the pullback in the following diagram:

$$\begin{array}{ccccc} \text{hom}(Y, A) & \xrightarrow{\text{hom}^c(i, p)} & \text{hom}(Y, B) \times_{\text{hom}(X, B)} \text{hom}(X, A) & \longrightarrow & \text{hom}(X, A) \\ & & \downarrow \lrcorner & & \downarrow \text{hom}(X, p) \\ & & \text{hom}(Y, B) & \xrightarrow{\text{hom}(i, B)} & \text{hom}(X, B). \end{array}$$

Of course, the right-hand vertical of this square is a left compcial fibration by the result of the last sentence, since it is the right corner closure of p and the inclusion $\emptyset \hookrightarrow X$. It follows that its left-hand vertical is also a left compcial fibration, since these are stable under pullback, whose codomain $\text{hom}(Y, B)$ is a weak left compcial set as already discussed. Consequently its domain, our pullback, is also a weak left compcial set so we may apply Corollary 49 to show that it is actually a weak compcial set which then enables us to apply the same corollary again to show that $\text{hom}^c(i, p)$ is a compcial fibration as required. \square

Corollary 76. *If $e : U \hookrightarrow V$ is an anodyne extension and $i : X \hookrightarrow Y$ is any inclusion then their corner tensor $e \otimes_c i$ has the LLP with respect to all compcial sets and all compcial fibrations between compcial sets.*

Proof. Apply Observation 122 to show that this result is simply dual to that of the last theorem under the adjunction $- \otimes_c i \dashv \text{hom}^c(i, *)$. \square

Observation 77 (*The Gray-category of weak complicial sets*). We may canonically enrich Strat with respect to its Gray tensor \otimes , to obtain an enriched category in which $\text{hom}(X, Y)$ is the stratified homset between stratified sets X and Y (see Kelly [13, p. 36] for the general construction of which this is a special case). Theorem 75 now tells us that the homsets of its enriched full subcategory Wcs of weak complicial sets are all themselves weak complicial sets. We call such (Strat, \otimes) -enriched categories whose homsets are weak complicial sets (*complicial*) *Gray-categories* and the reader may find out much more about these structures by consulting the companion paper [23].

5.3. A characterisation of strict complicial sets

Before moving on, these results allow us to establish another important characterisation of strict complicial sets amongst the weak ones.

Definition 78 (*Reprise of Definition 117 of [24]*). If X is a stratified set, we say that an n -simplex $x \in X$ is *pre-degenerate* at k if its face $x \cdot \alpha$ is thin whenever $\alpha : [m] \rightarrow [n]$ is a simplicial operator whose image contains the vertices $k, k+1 \in [n]$. Most importantly, the degeneracy condition on stratifications ensures that if x is degenerate at k (in the sense of Definition 2) then it is pre-degenerate at k .

Conversely, we say that X is *well-tempered* if whenever $x \in X$ is pre-degenerate at k then it is actually degenerate at k . The slogan here is that in a well-tempered stratified set, thinness is a sufficient property for the detection of degeneracy.

Lemma 79. *If Y is a well-tempered stratified set then every stratified map $h : X \otimes \Delta[1]_t \rightarrow Y$ factors through the projection map $\pi_X : X \otimes \Delta[1]_t \rightarrow X$.*

Proof (*essentially that of Corollary 164 of [24]*). It is clear that if h may be factored through π_X to give a stratified map $\hat{h} : X \rightarrow Y$ then this must be given by $\hat{h}(x) \stackrel{\text{def}}{=} h(x, 0)$, in which expression the symbol 0 represents the instance of the constant operator of Definition 3 whose dimension matches that of x . To show that this does indeed provide us with an appropriate factor we need to demonstrate that $h(x, \rho_{k+1}) = \hat{h}(x) = h(x, 0)$ for each one of the operators ρ_{k+1} of Definition 3.

For definiteness let r be the dimension of $x \in X$ and we will decorate the operators of Definition 3 (p. 1085) with their superscripted dimension. Consider the $(r+1)$ -simplex $(x \cdot \sigma_k, \rho_{k+1}^{r+1})$ of $X \otimes \Delta[1]_t$ and observe that it is pre-degenerate at k if and only if each one of its ordinates is pre-degenerate at k . However $x \cdot \sigma_k$ is certainly pre-degenerate at k , since it is degenerate there, and every simplex of dimension greater than 0 is thin in $\Delta[1]_t$ which clearly implies that every one of its $(r+1)$ -simplices is pre-degenerate at k . Now stratified maps clearly preserve pre-degeneracy so it follows that $h(x \cdot \sigma_k, \rho_{k+1}^{r+1})$ is pre-degenerate at k in Y and so there exists an r -simplex $y \in Y$ with $h(x \cdot \sigma_k, \rho_{k+1}^{r+1}) = y \cdot \sigma_k$ since Y is well-tempered. Observe now that we have the following calculations

$$y = (y \cdot \sigma_k) \cdot \delta_k = h(x \cdot \sigma_k, \rho_{k+1}^{r+1}) \cdot \delta_k = h((x \cdot \sigma_k) \cdot \delta_k, \rho_{k+1}^{r+1} \circ \delta_k) = h(x, \rho_k^r),$$

$$\begin{aligned}
 y &= (y \cdot \sigma_k) \cdot \delta_{k+1} = h(x \cdot \sigma_k, \rho_{k+1}^{r+1}) \cdot \delta_{k+1} \\
 &= h((x \cdot \sigma_k) \cdot \delta_{k+1}, \rho_{k+1}^{r+1} \circ \delta_{k+1}) = h(x, \rho_{k+1}^r)
 \end{aligned}$$

wherein we rely repeatedly on the simplicial identities $\sigma_k \circ \delta_k = \text{id} = \sigma_k \circ \delta_{k+1}$ and the easy observations that $\rho_{k+1}^{r+1} \circ \delta_k = \rho_k^r$ and $\rho_{k+1}^{r+1} \circ \delta_{k+1} = \rho_{k+1}^r$. In other words we have shown that for each $k = 0, 1, \dots, r$ we have $h(x, \rho_k^r) = h(x, \rho_{k+1}^r)$ and composing these equalities we find that $h(x, \rho_k^r) = h(x, \rho_{r+1}^r) = h(x, 0)$ as required where the last equality simply expresses the fact that the operators ρ_{r+1}^r and 0 are identical. \square

Theorem 80. *A stratified set A is a (strict) complicial set if and only if it is a weak complicial set and it is well-tempered.*

Proof. The “only if” direction follows immediately by the arguments of Example 17 and Lemma 163 of [24].

To prove the converse, first observe that it is enough to show that if A is well-tempered and a weak complicial set then it has unique fillers for inner complicial horns. So suppose that we are given a stratified map $f : \Delta^k[n] \rightarrow A$ along with a pair of extensions $k_0, k_1 : \Delta^k[n] \rightarrow A$ of that

map along the inclusion $\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$. From this information we may construct a stratified map $h : (\Delta^k[n] \otimes \Delta[1]_t) \cup (\Delta^k[n] \otimes \partial\Delta[1]) \rightarrow A$ by letting $h(\alpha, \beta) \stackrel{\text{def}}{=} f(\alpha)$ on $\Delta^k[n] \otimes \Delta[1]_t$ and letting $h(\alpha, 0) \stackrel{\text{def}}{=} k_0(\alpha)$ and $h(\alpha, 1) \stackrel{\text{def}}{=} k_1(\alpha)$ on $\Delta^k[n] \otimes \partial\Delta[1]$, where 0 and 1 denote appropriate instances of the constant operators given in Definition 3. Of course each of these pieces of h is stratified and they match where mutually defined, because k_0 and k_1 both extend f , thus demonstrating that it is a well-defined stratified map. Furthermore, it may be extended to a stratified map $\tilde{h} : \Delta^k[n] \otimes \Delta[1]_t \rightarrow A$, because A is a weak complicial set and the inclusion of the domain of h into $\Delta^k[n] \otimes \Delta[1]_t$ is the corner tensor of the inner horn inclusion $\Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ and the inclusion $\partial\Delta[1] \xrightarrow{\subseteq_r} \Delta[1]_t$ which is an (inner) anodyne extension by Observation 74. Since A is well-tempered we may now apply Lemma 79 and factor \tilde{h} through the projection $\pi_{\Delta^k[n]} : \Delta^k[n] \otimes \Delta[1]_t \rightarrow \Delta^k[n]$ to give a map $\hat{h} : \Delta^k[n] \rightarrow A$ and now we find that $k_0(\alpha) = f(\alpha, 0) = \tilde{f}(\alpha, 0) = \hat{f}(\alpha)$ and $k_1(\alpha) = f(\alpha, 1) = \tilde{f}(\alpha, 1) = \hat{f}(\alpha)$, which demonstrates that $k_0 = k_1$ and thus that f has exactly one extension as required. \square

6. Quillen model structures on stratified sets

In this section we muster the machinery developed in the last few sections to demonstrate that the category of stratified sets Strat supports a natural Quillen model structure whose fibrant objects are precisely the weak complicial sets. We do so using Jeffery Smith’s theorem for locally presentable categories, the conditions of which we have recounted as Theorem 125 in Appendix A. As discussed in Observation 12, the category Strat is locally finitely presentable and thus provides a context within which to apply this theorem.

Definition 81. We define I to be the set of boundary and thin simplex inclusions

$$\{\partial\Delta[n] \xrightarrow{\subseteq_r} \Delta[n] \mid n = 0, 1, \dots\} \cup \{\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_t \mid n = 1, 2, \dots\}$$

whose cellular completion $\text{cell}(I)$ (cf. Notation 20, p. 1091) is the class of all inclusions of stratified sets (cf. Example 21, p. 1091). Consequently, the members of the corresponding class of fibrations $\text{fib}(I)$, called *trivial fibrations*, all enjoy the RLP (cf. Notation 20, p. 1091) with respect to arbitrary inclusions of stratified sets. Notice that it is immediate that all trivial fibrations are also complicial fibrations (cf. Definition 22, p. 1091).

6.1. Homotopy equivalences of weak complicial sets

The next few definitions and results are appropriated, with appropriate modifications, from classical simplicial homotopy theory.

Definition 82. If $f, g : X \rightarrow Y$ are stratified maps then a *simple homotopy* from f to g is a stratified map $h : X \otimes \Delta[1]_t \rightarrow Y$ for which $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in X$. Notice that in order to make sense of these expressions we assume that 0 and 1 denote suitable instances of the constant operators introduced in Notation 3.

We write $f \sim_1 g$ if there exists a simple homotopy from f to g and let $f \sim g$, the *homotopy relation*, denote the transitive closure of that relation.

Observation 83. Taking duals under the adjunction $X \otimes - \dashv \text{hom}(X, *)$ and appealing to Yoneda's lemma we see that a simple homotopy corresponds to a thin 1-simplex \hat{h} in the stratified set $\text{hom}(X, Y)$ whose vertices are $f = \hat{h} \cdot \varepsilon_0$ and $g = \hat{h} \cdot \varepsilon_1$. This presentation immediately tells us that the simple homotopy relation \sim_1 is already transitive (and is thus identical to \sim) whenever the codomain of our maps is a weak complicial set A . To verify this fact simply observe that, by Theorem 75, $\text{hom}(X, A)$ is a weak complicial set whenever A is and demonstrate transitivity of simple homotopy using fillers for suitable (thin) 1-dimensional horns in $\text{hom}(X, A)$ to compose the witnessing simple homotopies.

Definition 84. If X and Y are stratified sets then a *homotopy equivalence* between them is a stratified map $e : X \rightarrow Y$ which has an *equivalence inverse* $e' : Y \rightarrow X$ for which we have $e' \circ e \sim \text{id}_X$ and $e \circ e' \sim \text{id}_Y$.

Observation 85. The homotopy relation is preserved by pre-composition and post-composition in Strat, so we may form a *homotopy category* $\Pi(\text{Strat})$ by taking the quotient of each of the homsets of Strat under the homotopy relation. Then a stratified map $e : X \rightarrow Y$ is a homotopy equivalence if and only if its corresponding homotopy class $[e]_{\sim} : X \rightarrow Y$ is an isomorphism in $\Pi(\text{Strat})$.

Using this observation we may immediately derive many useful properties of homotopy equivalences directly from the corresponding facts about isomorphisms in any category. In particular, in the sequel we will make use of the following very simple observations:

- **Homotopy stability** If e is a homotopy equivalence then so is any stratified map homotopic to e .
- **2-of-3 property** If two of the stratified maps e , f and their composite $f \circ e$ are homotopy equivalences then so is the third.
- **Stability under retract** Retracts of homotopy equivalences are again homotopy equivalences.

- **Left inverse property** If $e : X \rightarrow Y$ is a homotopy equivalence and $\bar{e} : Y \rightarrow X$ is such that $\bar{e} \circ e \sim \text{id}_X$ (left equivalence inverse) then we also have $e \circ \bar{e} \sim \text{id}_Y$ (right equivalence inverse).

Observation 86. The Gray-category $\underline{\mathbf{Wcs}}$ of weak complicial sets, which we introduced in Observation 77, gives rise to a Kan complex enriched category by applying the Gray tensor product preserving 0-superstructure functor sp_0 (cf. Notation 13, p. 1088) to its homsets. We may apply Cordier and Porter's homotopy coherent nerve functor [3] to this structure to obtain a quasi-category $N_{hc}(\underline{\mathbf{Wcs}})$. A presentation of this nerve construction suited to our needs here is provided in the companion paper [23], which generalises the classical homotopy coherent nerve construction to provide a faithful embedding of the category of Gray-categories into the category of weak complicial sets.

Now Definition 84 above may simply be regarded as saying that the stratified map $e : A \rightarrow B$ has an equivalence inverse in $N_{hc}(\underline{\mathbf{Wcs}})$ in the sense of Theorem 56. So applying that theorem, it follows that any homotopy equivalence gives rise to a simplicial map $E_3^- \rightarrow N_{hc}(\underline{\mathbf{Wcs}})$ which may be pictured as:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & B & \xrightarrow{e'} & A \\
 & \nearrow e & \downarrow h \simeq & \searrow e & \\
 A & & & & B \\
 & \searrow e & \downarrow = & \nearrow e & \\
 & & A & \xrightarrow{e} & B
 \end{array}
 & \xrightarrow[t \simeq]{} &
 \begin{array}{ccccc}
 & & B & \xrightarrow{e'} & A \\
 & \nearrow e & \searrow k \simeq & \searrow e & \\
 A & & & & B \\
 & \searrow e & \downarrow = & \nearrow e & \\
 & & A & \xrightarrow{e} & B
 \end{array}
 \end{array}$$

Unwinding the definition of $N_{hc}(\underline{\mathbf{Wcs}})$ given in [23] it is easily seen that this data amounts to a choice of simple homotopies

$$h : A \otimes \Delta[1]_t \rightarrow A \quad \text{with } h(a, 0) = e'(e(a)) \text{ and } h(a, 1) = a$$

$$k : B \otimes \Delta[1]_t \rightarrow B \quad \text{with } k(b, 0) = e(e'(b)) \text{ and } k(b, 1) = b$$

and a “double” homotopy

$$\begin{aligned}
 t : A \otimes \Delta[1]_t \otimes \Delta[1]_t &\rightarrow B && \text{with } t(a, 0, \beta) = e(h(a, \beta)) \\
 &&& \text{and } t(a, \alpha, 0) = k(e(a), \alpha) \\
 &&& \text{and } t(a, \alpha, 1) = t(a, 1, \beta) = e(a)
 \end{aligned}$$

connecting them.

Definition 87. We say that a stratified map $e : X \rightarrow Y$ is a (simple) deformation retraction if there is a stratified map $m : X \rightarrow Y$ with $e \circ m = \text{id}_Y$ and a simple homotopy $d : A \otimes \Delta[1]_t \rightarrow A$ from $m \circ e$ to id_A with $e(d(a, \alpha)) = e(a)$ (for all $a \in A$ and $\alpha \in \Delta[1]_t$).

Lemma 88. If B is a weak complicial set and $e : A \rightarrow B$ is a complicial fibration then the following are equivalent:

- e is a homotopy equivalence,

- (ii) e is a deformation retraction, and
- (iii) e is a trivial fibration.

Observation 89. Notice that we need not assume explicitly that A is a weak complicial set because we may immediately infer that this is the case from the compliciality assumptions on B and e and the fact that the class of complicial fibrations is closed under composition. In future in these cases we will simply say that such a map is a *complicial fibration of weak complicial sets*.

Proof (of Lemma 88). This is fundamentally a classical result. Clearly a deformation retraction is a special sort of homotopy equivalence, so the implication (ii) \Rightarrow (i) is trivial. The implication (iii) \Rightarrow (ii) is also routine, we simply use the trivial fibration assumption on e to make successive lifts

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & A \\
 \downarrow & \nearrow \exists m & \downarrow e \\
 B & \xrightarrow{\text{id}_B} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A + A & \xrightarrow{(m \circ e, \text{id}_A)} & A \\
 \downarrow & \nearrow \exists d & \downarrow e \\
 A \otimes \Delta[1]_t & \xrightarrow{e \circ \pi_A} & B
 \end{array}$$

to construct m and d satisfying the properties required by the last definition.

To prove the reverse implication (i) \Rightarrow (ii), assume that e is an equivalence with inverse e' and that we are given the simple homotopies h and k and the double homotopy t described in Observation 86. Now consider the squares

$$\begin{array}{ccc}
 B \otimes \Lambda^0[1] & \xrightarrow{e' \circ \pi_B} & A \\
 \downarrow \subseteq_r & \nearrow \exists \bar{k} & \downarrow e \\
 B \otimes \Delta[1]_t & \xrightarrow{k} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{f} & A \\
 \downarrow \subseteq_r & \nearrow \exists \bar{t} & \downarrow e \\
 A \otimes \Delta[1]_t \otimes \Delta[1]_t & \xrightarrow{t} & B
 \end{array}
 \tag{10}$$

wherein the left hand vertical of the left hand square is the corner tensor

$$(\emptyset \xrightarrow{\subseteq_r} B) \otimes_c (\Lambda^0[1] \xrightarrow{\subseteq_r} \Delta[1]_t)$$

and the stratified set X is defined to be the regular subset which makes the left-hand vertical of the right-hand square into the corner tensor:

$$(\emptyset \xrightarrow{\subseteq_r} A) \otimes_c (\Lambda^0[1] \xrightarrow{\subseteq_r} \Delta[1]_t) \otimes_c (\partial \Delta[1] \xrightarrow{\subseteq_r} \Delta[1]_t).$$

These are both corner tensors of an inclusion with the elementary anodyne extension $\Lambda^0[1] \xrightarrow{\subseteq_r} \Delta[1]_t$ so we may apply Corollary 76 to show that they both have the LLP with respect to the complicial fibration $e: A \rightarrow B$. This fact explains the existence of the lift \bar{k} in the left hand square of display (10), which we may use to define a stratified map $m: B \rightarrow A$ by $m(b) \stackrel{\text{def}}{=} \bar{k}(b, 1)$. Furthermore, the commutativity of its upper triangle tells us that \bar{k} is a simple

homotopy from e' to m whereas its lower triangle tells us that $e(m(b)) = e(\bar{k}(b, 1)) = k(b, 1) = b$ (amongst other things).

Now we may turn to the right-hand square of display (10), and proceed to define the stratified map f , whose domain X is the stratified set $A \otimes ((\Delta[1]_t \otimes \partial\Delta[1]) \cup (\Lambda^0[1] \otimes \Delta[1]_t))$. This splits naturally into three components each of which is isomorphic to $A \otimes \Delta[1]_t$ and upon which we define f in a piecewise manner:

$$f(a, \alpha, 0) \stackrel{\text{def}}{=} \bar{k}(e(a), \alpha), \quad f(a, \alpha, 1) \stackrel{\text{def}}{=} a, \quad f(a, 0, \beta) \stackrel{\text{def}}{=} h(a, \beta).$$

In other words, the first two clauses specify how f acts on the disjoint components of $A \otimes \Delta[1]_t \otimes \partial\Delta[1]$ and the last one specifies how it acts on $A \otimes \Lambda^0[1] \otimes \Delta[1]_t$. To check that f is well defined it is enough to observe that the pieces of its definition match at the “corners” where they meet, since $\bar{k}(e(a), 0) = e'(e(a)) = h(a, 0)$ and $h(a, 1) = a$, and that it respects the stratification on each component (since \bar{k} , e and h are all stratified maps). Furthermore, comparing the definition of f with the properties of the boundary of t laid out at the bottom of Observation 86 and applying the defining property $e \circ \bar{k} = k$ of \bar{k} it is now easily seen that the right-hand square in display (10) commutes and thus that the lift \bar{t} exists as advertised there.

Finally, it remains to define the simple homotopy $d: A \otimes \Delta[1]_t \rightarrow A$ by letting $d(a, \beta) \stackrel{\text{def}}{=} \bar{t}(a, 1, \beta)$. Applying the commutativities of the triangles in the right hand square of display (10) and the properties of t given in Observation 86 we discover that

$$\begin{aligned} d(a, 0) &= \bar{t}(a, 1, 0) = f(a, 1, 0) = \bar{k}(e(a), 1) = m(e(a)), \\ d(a, 1) &= \bar{t}(a, 1, 1) = f(a, 1, 1) = a, \\ e(d(a, \beta)) &= e(\bar{t}(a, 1, \beta)) = t(a, 1, \beta) = e(a) \end{aligned}$$

thus verifying that d completes the data required to demonstrate that e is a deformation retraction as required.

All that remains for us is to prove (ii) \Rightarrow (iii), so suppose that e is a deformation retraction witnessed by the stratified map m and the simple homotopy d of Definition 87 and consider the lifting problem depicted in the left-hand square below:

$$\begin{array}{ccc} U & \xrightarrow{f} & A \\ \subseteq_s \downarrow & & \downarrow e \\ V & \xrightarrow{g} & B \end{array} \quad \begin{array}{ccc} (U \otimes \Delta[1]_t) \cup (V \otimes \Lambda^0[1]) & \xrightarrow{\bar{f}} & A \\ \subseteq_s \downarrow & \nearrow \exists \bar{h} & \downarrow e \\ V \otimes \Delta[1]_t & \xrightarrow{g \circ \pi_V} & B. \end{array} \quad (11)$$

We provide a solution to this problem by constructing the right-hand square, in which \bar{f} is defined in a piecewise manner:

$$\bar{f}(u, \alpha) \stackrel{\text{def}}{=} d(f(u), \alpha) \quad \text{on } U \otimes \Delta[1]_t \quad \text{and} \quad \bar{f}(v, 0) \stackrel{\text{def}}{=} m(g(v)) \quad \text{on } V \otimes \Lambda^0[1].$$

This is well defined because the actions on these components respect stratifications (since d , m , f and g are all stratified maps) and they match at the intersection of their domains where $d(f(u), 0) = m(e(f(u))) = m(g(u))$, in which the former equality holds because d is a simple

homotopy from $m \circ e$ to id_A and the latter one simply follows from the commutativity of the original lifting problem. Notice now that using the properties of e , m and d as the components of a deformation retraction and the commutativity of our original lifting problem again we have

$$\begin{aligned} e(\bar{f}(u, \alpha)) &= e(d(f(u), \alpha)) = e(f(u)) = g(u), \\ e(\bar{f}(v, 0)) &= e(m(g(v))) = g(v) \end{aligned}$$

or, in other words, the square displayed does indeed commute. Furthermore its left hand vertical is the corner tensor

$$(U \xrightarrow{\subseteq_s} V) \otimes_c (\Lambda^0[1] \xrightarrow{\subseteq_r} \Delta[1]_r)$$

which has the LLP with respect to the complicial fibration e by Corollary 76 and so the lift \bar{h} (dotted arrow) exists as depicted in the right hand square of display (11). Now it is trivially verified, directly from the properties of \bar{h} as the stated lift, that the stratified map $\bar{g}: V \rightarrow A$ defined by $\bar{g}(v) = \bar{h}(v, 1)$ is the required solution to the left hand lifting problem. Ultimately, it follows that e is a trivial fibration since, in particular, we have shown that it has the RLP with respect to all boundary and thin simplex inclusions. \square

Observation 90. The proofs of the implications (iii) \Rightarrow (ii) and (ii) \Rightarrow (i) in the last lemma were independent of any weak compliciality assumptions, so we may infer that **any** trivial fibration between arbitrary stratified sets is a deformation retraction and that these in turn are homotopy equivalences.

6.2. Weak equivalences of stratified sets

Definition 91. For this subsection and the next we will do everything relative to a fixed (small) set of inclusions J in Strat, which we assume satisfies the condition

- (i) each elementary anodyne extension is an element of J

thereby ensuring that every J -fibrant object (cf. Notation 20, p. 1091) is a weak complicial set.

We say that a stratified map $w: X \rightarrow Y$ is a J -weak equivalence if and only if the associated stratified map $\text{hom}(w, A): \text{hom}(Y, A) \rightarrow \text{hom}(X, A)$ is a homotopy equivalence for each J -fibrant stratified set A and we let \mathcal{W}_J denote the class of all J -weak equivalences in Strat. Unless otherwise stated, we will generally also assume that our set J satisfies the condition

- (ii) each element of J is a J -weak equivalence

which postulates a stability property closely related to the result established in Corollary 76 for anodyne extensions.

The construction to follow provides a Quillen model structure whose fibrant objects are the J -fibrant stratified sets and whose fibrations between fibrant objects are precisely the J -fibrations between those stratified sets. This will allow us to construct model structures whose fibrant objects are weak complicial sets, n -trivial (cf. Notation 13, p. 1088) weak complicial sets, quasi-categories under their standard stratification and so forth.

To make our nomenclature match with that of previous sections we shall call the J -fibrant objects J -weak complicial sets, the J -fibrations J -complicial fibrations, the J -cell complexes J -anodyne extensions and so on. Also let \mathbf{Wcs}_J denote the full subcategory of \mathbf{Strat} whose objects are the J -weak complicial sets.

Lemma 92. *Suppose that J is a small set of stratified inclusions that satisfies condition (i) of Definition 91 and suppose that $e : U \hookrightarrow V$ is an inclusion of stratified sets then the following are equivalent:*

- (i) e is a J -weak equivalence,
- (ii) $\mathrm{hom}(e, A) : \mathrm{hom}(V, A) \rightarrow \mathrm{hom}(U, A)$ is a trivial fibration for all J -weak complicial sets A , and
- (iii) for all inclusions $i : X \hookrightarrow Y$ the corner tensor $e \otimes_c i$ has the LLP with respect to each J -weak complicial set A .

Proof. Every J -weak complicial set is, in particular, a weak complicial set so we may apply Theorem 75 to the inclusion e to show that $\mathrm{hom}(e, A)$ is a complicial fibration of weak complicial sets whenever A is a J -weak complicial set. Consequently, applying Lemma 88 we see that $\mathrm{hom}(e, A)$ is a homotopy equivalence if and only if it is a trivial fibration. So, quantifying over all J -weak complicial sets and applying the J -weak equivalence definition, we have established the equivalence (i) \Leftrightarrow (ii). The remaining equivalence follows routinely by applying Observation 122 to the adjunction $e \otimes_c - \dashv \mathrm{hom}^c(e, *)$. \square

Example 93. Recasting the result of Corollary 76 using Observation 122 and applying the last lemma we may verify that the countable set

$$J_c \stackrel{\mathrm{def}}{=} \left\{ \Delta^k[n] \xrightarrow{\subseteq_r} \Delta^k[n] \mid n = 1, 2, \dots \text{ and } 0 \leq k \leq n \right\} \\ \cup \left\{ \Delta^k[n]' \xrightarrow{\subseteq_e} \Delta^k[n]'' \mid n = 2, \dots \text{ and } 0 \leq k \leq n \right\}$$

of all elementary anodyne extensions provides a *minimal* set satisfying the conditions given in Definition 91.

Observation 94 (*Homotopy equivalence implies J -weak equivalence*). The contravariant functor $\mathrm{hom}(*, A)$ has a canonical enrichment whose action on homsets $\mathrm{hom}(X, Y) \rightarrow \mathrm{hom}(\mathrm{hom}(Y, A), \mathrm{hom}(X, A))$ is formed by taking the dual of the composition $\circ : \mathrm{hom}(Y, A) \otimes \mathrm{hom}(X, Y) \rightarrow \mathrm{hom}(X, A)$ of \mathbf{Strat}_\otimes under the appropriate closure adjunction. In this way, for each weak complicial set A we obtain a \mathbf{Strat}_\otimes -enriched functor $\mathrm{hom}(*, A) : \mathbf{Strat}_\otimes^{\mathrm{op}} \rightarrow \mathbf{Wcs}$ which carries the thin 1-simplices in the homsets of \mathbf{Strat}_\otimes , that is to say simple homotopies, to thin 1-simplices in the homsets of \mathbf{Wcs} and thus preserves the homotopy relation \sim between stratified maps. Consequently it maps left (resp. right) homotopy inverses to right (resp. left) homotopy inverses and therefore preserves homotopy equivalences, thus demonstrating that any homotopy equivalence of stratified sets is a J -weak equivalence.

Observation 95 (*A partial converse*). Suppose that $w : A \rightarrow B$ is a J -weak equivalence between J -weak complicial sets then, since A is a J -weak complicial set, we find that the

stratified map $\text{hom}(w, A) : \text{hom}(B, A) \rightarrow \text{hom}(A, A)$ has a homotopy inverse $\bar{w} : \text{hom}(A, A) \rightarrow \text{hom}(B, A)$ for which the right inverse homotopy $\text{hom}(w, A) \circ \bar{w} \sim \text{id}_{\text{hom}(A, A)}$ may be witnessed by a simple homotopy $\bar{h} : \text{hom}(A, A) \otimes \Delta[1]_I \rightarrow \text{hom}(A, A)$. Consequently, if we define maps $w' : B \rightarrow A$ by $w' = \bar{w}(\text{id}_A)$ and $h : A \otimes \Delta[1]_I \rightarrow A$ by $h(a, \alpha) = \bar{h}(\text{id}_A \cdot \eta, \alpha)(a)$ then we have

$$h(a, 0) = \bar{h}(\text{id}_A, 0)(a) = \text{hom}(w, A)(\bar{w}(\text{id}_A))(a) = w'(w(a)),$$

$$h(a, 1) = \bar{h}(\text{id}_A, 1)(a) = \text{id}_{\text{hom}(A, A)}(\text{id}_A)(a) = a$$

or, in other words, h is a simple homotopy from $w' \circ w$ to id_A .

Applying $\text{hom}(-, B)$ to this, and consulting the last observation, we obtain a simple homotopy from $\text{hom}(w, B) \circ \text{hom}(w', B)$ to $\text{id}_{\text{hom}(B, B)}$ thus demonstrating that $\text{hom}(w', B)$ is a right equivalence inverse of $\text{hom}(w, B)$. This latter map is, however, a homotopy equivalence, since B is a J -weak complicial set and w is a J -weak equivalence, so it follows that $\text{hom}(w', B)$ is also a left equivalence inverse of $\text{hom}(w, B)$ by Observation 85. Finally, applying the argument used above to obtain h from \bar{h} to the resulting simple homotopy \bar{k} from $\text{hom}(w', B) \circ \text{hom}(w, B)$ to $\text{id}_{\text{hom}(B, B)}$ we obtain a simple homotopy k from $w \circ w'$ to id_B thus completing the demonstration that w is a homotopy equivalence.

Observation 96. Fix an inclusion $i : X \hookrightarrow Y$ and observe that we may apply clause (iii) of Lemma 92 and the fact that every element of J is both a J -weak equivalence and an inclusion (under the assumptions of Definition 91) to show that for each $e \in J$ the corner tensor $e \otimes_c i$ has the LLP with respect to each J -weak complicial set A . Applying Observation 122 to the adjunction $- \otimes_c i \dashv \text{hom}^c(i, *)$, we find that this is equivalent to saying that $\text{hom}(i, A) : \text{hom}(Y, A) \rightarrow \text{hom}(X, A)$ has the RLP with respect to each inclusion in J and that it is thus a J -complicial fibration.

Applying this result to the (unique) inclusion $! : \emptyset \hookrightarrow X$ whose domain is the empty stratified set, we find that whenever A is a J -weak complicial set the (also unique) stratified map $\text{hom}(!, A) : \text{hom}(X, A) \rightarrow \text{hom}(\emptyset, A) \cong 1$ is a J -complicial fibration. In other words, in that circumstance $\text{hom}(X, A)$ is also a J -weak complicial set.

Observation 97 (*Verifying the conditions of Jeffery Smith's theorem*). Our intention is to show that the set of inclusions I and the class of J -weak equivalences satisfy the conditions of Theorem 125. Indeed, we are already in a position to verify the first three clauses of its statement.

- (1) Observation 85 tells us that the class of homotopy equivalences is closed under retracts and enjoys the 2-of-3 property. It is thus clear, directly from Definition 91 and the functoriality of $\text{hom}(*, A)$, that the class of J -weak equivalences \mathcal{W}_J also possesses these properties.
- (2) From Observation 90 we know that all I -fibrations (trivial fibrations) are homotopy equivalences and thus that they are all J -weak equivalences by Observation 94.
- (3) Example 21 tells us that the class of all inclusions of stratified sets is closed under pushout, transfinite composition and retraction and also that it is equal to $\text{cof}(I)$. Lemma 92 reveals that an inclusion $e : U \hookrightarrow V$ is in \mathcal{W}_J if and only if $\text{hom}(e, A) : \text{hom}(V, A) \rightarrow \text{hom}(U, A)$ is a trivial fibration for each J -weak complicial set A . However $\text{hom}(*, A)$ carries transfinite composites and pushouts to the corresponding limits in **Strat** and any class of the form $\text{fib}(I)$, such as the class of trivial fibrations, is closed under those limits. Combining these facts, we see that if $e : U \hookrightarrow V$ is a transfinite composite of pushouts of elements of $\text{cof}(I) \cap \mathcal{W}_J$ then

$\text{hom}(e, A) : \text{hom}(V, A) \rightarrow \text{hom}(U, A)$ is a corresponding transfinite limit of pullbacks of trivial fibrations which is thus itself a trivial fibration. That however implies that $\text{hom}(e, A)$ is a trivial fibration for each J -weak complicial set A and thus that the inclusion e is a J -weak equivalence as required.

Under our assumption that the elements of J are all inclusions, condition (ii) of Definition 91 simply states that the set J is a subset of the class $\text{cof}(I) \cap \mathcal{W}_J$. However, the last of the properties above verifies that this latter class is closed under the operations used to derive the class of J -anodyne extensions from J itself. It follows, therefore, that all J -anodyne extensions are also J -weak equivalences.

We demonstrate the fourth condition of Theorem 125 by showing that the class of J -weak equivalences is actually an *accessible class* of maps (cf. Observation 124, p. 1148 and Beke [2]), which we do in two steps:

Observation 98 (*The class of trivial fibrations is accessible*). By Observation 12 we know that the category Strat is locally finitely presentable. Furthermore we may argue, as in Observation 123, that the full subcategory TFib of Strat² whose objects are the trivial fibrations is simply the injectivity class associated with the set of squares of the form:

$$\begin{array}{ccc} \partial \Delta[n] & \xrightarrow{\subseteq_r} & \Delta[n] \\ \subseteq_r \downarrow & & \downarrow \text{id} \\ \Delta[n] & \xrightarrow{\text{id}} & \Delta[n] \end{array} \quad \begin{array}{ccc} \Delta[n] & \xrightarrow{\subseteq_e} & \Delta[n]_t \\ \subseteq_e \downarrow & & \downarrow \text{id} \\ \Delta[n]_t & \xrightarrow{\text{id}} & \Delta[n]_t. \end{array}$$

It follows that we may apply Observation 124 to show that the class of trivial fibrations is an accessible class of maps in Strat. Indeed, with a little more work we may show that TFib is \aleph_1 -accessible and \aleph_0 -accessibly embedded in Strat², although we will not need that result here.

Observation 99 (*The class of J -weak equivalences is accessible*). In a similar fashion we may describe the class of J -complicial fibrations between J -weak complicial sets as an injectivity class CFib _{J} in Strat². To be precise the objects of this subcategory are the morphisms which are injective with respect to the squares of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ U_j & \xrightarrow{j} & V_j \end{array} \quad \begin{array}{ccc} U_j & \xrightarrow{j} & V_j \\ j \downarrow & & \downarrow \text{id} \\ V_j & \xrightarrow{\text{id}} & V_j \end{array} \quad (12)$$

for each $j : U_j \hookrightarrow V_j$ in J . Here injectivity with respect to squares of the left-hand form ensures that the codomains of morphisms in our class are J -weak complicial sets and injectivity with respect to squares of the right-hand form ensures that these morphisms are themselves J -complicial fibrations. We will use K to denote the set of those morphisms of Strat² depicted in the above display.

Now we may apply Observation 123 to the locally finitely presentable category Strat² and the set K of its squares to obtain an accessible weak reflection of Strat² into CFib _{J} . We denote the

weak reflection of an object $f: X \rightarrow Y$ of \mathbf{Strat}^2 into \mathbf{CFib}_J by $f^*: X_f^* \rightarrow Y_f^*$ and the use the notation

$$\begin{array}{ccc} X & \xrightarrow{\eta_f^d} & X_f^* \\ f \downarrow & & \downarrow f^* \\ Y & \xrightarrow{\eta_f^c} & Y_f^* \end{array} \quad (13)$$

for the associated component of the unit of this weak reflection. This is simply the component of the defining colimiting cone of $(-)^*$ from the first element $\text{id}_{\mathbf{Strat}^2}$ of the chain constructed in Observation 123. Since all colimits in \mathbf{Strat}^2 and its endo-functor category are constructed pointwise in \mathbf{Strat} , we know that the maps η_f^d and η_f^c are J -anodyne extensions because they are constructed as transfinite composites of pushouts of coproducts of the horizontal maps in the squares in display (12), each of which is an element of J or an identity. Now J -anodyne extensions are J -weak equivalences, by Observation 97, so we may apply the 2-of-3 property for these to the square in display (13) to show that f is a J -weak equivalence if and only if f^* is a J -weak equivalence. However, since f^* is a J -complicial fibration of J -weak complicial sets we may apply Observation 95 and Lemma 88 to show that f^* is a J -weak equivalence if and only if it is a trivial fibration.

In summary, we have constructed an accessible endo-functor $(-)^*$ of \mathbf{Strat}^2 with the property that a stratified map $f: X \rightarrow Y$ is an object of \mathbf{WEqv}_J , the full subcategory of \mathbf{Strat}^2 whose objects are J -weak equivalences, if and only if its weak reflection $f^*: X_f^* \rightarrow Y_f^*$ is an object of the accessible and accessibly embedded full subcategory \mathbf{TFib} of trivial fibrations. In other words, \mathbf{WEqv}_J is a pseudo-pullback of \mathbf{TFib} along the endo functor $(-)^*$ and consequently we may apply Theorem 5.1.6 of [14] to show that it too is an accessibly embedded, accessible subcategory of \mathbf{Strat}^2 . In this way we have shown that the J -weak equivalences form an accessible class, which is finally all we need in order to apply Jeffery Smith's theorem and establish the next theorem.

6.3. The J -complicial model structure

Theorem 100. *Each set of stratified inclusions J satisfying the conditions given in Definition 91 gives rise to a cofibrantly generated Quillen model structure on the category \mathbf{Strat} of stratified sets, called the J -complicial model structure, whose:*

- **weak equivalences** are the J -weak equivalences of Observation 91,
- **cofibrations** are simply inclusions of stratified sets, and whose
- **fibrant objects** are the J -weak complicial sets.

Proof. Apply Jeffery Smith's Theorem 125, using our Observations 97 and 99 to verify the required properties of I and \mathcal{W} . The proof that the fibrant objects in this model structure are exactly the J -weak complicial sets is postponed to Lemma 105 below. \square

Notation 101. We call the model structure derived in the last theorem the J -complicial model structure and refer to its trivial cofibrations as J -complicial cofibrations. Observation 97 tells us

that all J -anodyne extensions are J -complicial cofibrations, but there is no reason in general to believe that these classes coincide.

We also call the fibrations of this model structure *completely J -complicial fibrations*. The dual of our last observation is that every completely J -complicial fibration is a J -complicial fibration but that these classes may not coincide.

Of course, we may apply the general theory of Quillen model categories to our current situation and show that a stratified map is a J -complicial cofibration if and only if it has the LLP with respect to all completely J -complicial fibrations (and indeed vice versa), and it follows therefore that the class of such things is closed under pushout, transfinite composition and retraction.

If we omit the prefix “ J -” altogether then we will implicitly assume that we are working relative the set J_c discussed in Example 93. So the fibrant objects of the complicial model structure are precisely the weak complicial sets.

Observation 102. If $J \subseteq J'$ are two sets of stratified inclusions satisfying the conditions of Definition 91 then the corresponding complicial model structures share the same sets of cofibrations but differ in their sets of weak equivalences and completely complicial fibrations. We know, however, that every J -weak equivalence is also a J' -weak equivalence, which implies a corresponding relationship between respective classes of complicial cofibrations. Dually, it immediately follows that every (complete) J' -complicial fibration is a (complete) J -complicial fibration.

Observation 103 (*Localising an existing complicial model structure*). If we start with a set of inclusions J which satisfies the conditions of Definition 91 and K is any other set of inclusions then a $J \cup K$ -fibrant object is also J -fibrant so it follows that any J -weak equivalence is also a $J \cup K$ -weak equivalence. Consequently every element of J is a $J \cup K$ -weak equivalence so to show that $J \cup K$ satisfies condition (ii) of Definition 91 all we need do is check that each $k \in K$ is a $J \cup K$ -weak equivalence. However, since $J \cup K$ satisfies condition (i) of Definition 91 we may do this using one of the equivalent characterisations of Lemma 92.

It is a standard, and easily demonstrated, result of Quillen model category theory that a map $p : A \rightarrow B$ between fibrant objects is a fibration (resp. trivial fibration) if and only if it has the RLP with respect to all trivial cofibrations (resp. cofibrations) $i : X \rightarrow Y$ for which X and Y are fibrant. Now we know, by assumption and Theorem 100, that J gives rise to a complicial model structure and that any $J \cup K$ -weak complicial set A is a J -weak complicial set, so we may apply Observation 96 to show that any $\text{hom}(X, A)$ is also a J -weak complicial set. In particular, if $k : U_k \hookrightarrow V_k$ is an element of K then it follows that the domain and codomain of the stratified map $\text{hom}(k, A) : \text{hom}(V_k, A) \rightarrow \text{hom}(U_k, A)$ of clause (ii) in Lemma 92 are J -weak complicial sets and thus that we may apply the result recalled in the first sentence to show that $\text{hom}(k, A)$ is a trivial fibration if and only if it has the RLP with respect to all inclusions of J -weak complicial sets. Equivalently, it follows that we may show that k is a $J \cup K$ -weak equivalence by showing that it satisfies clause (iii) in Lemma 92 for those inclusions in this restricted class.

Example 104 (*The n -trivial complicial model structure*). To obtain a Quillen model structure whose fibrant objects are the n -trivial weak complicial sets, define the set

$$J_n \stackrel{\text{def}}{=} J_c \cup \{ \Delta[r] \xrightarrow{\subseteq_e} \Delta[r]_t \mid r \in \mathbb{N} \wedge r > n \}$$

for which the J_n -fibrant objects are those stratified sets with the desired properties. Now the n -superstructure functor sp_n (cf. Notation 13, p. 1088) acts as the identity on n -trivial stratified sets and we know, by Lemma 25, that it preserves weak compliciality. Combining that observation with Lemma 92 we find that $t_r \stackrel{\mathrm{def}}{=} \Delta[r] \xrightarrow{\subseteq_r} \Delta[r]_t$ ($r > n$) is a J_n -weak equivalence if and only if its corner tensor $t_r \otimes_c i$ with each inclusion $X \xrightarrow{i} Y$ has the LLP with respect to $\mathrm{sp}_n(A)$ for each weak complicial set A . Taking duals under the adjunction $\mathrm{th}_n \dashv \mathrm{sp}_n$ (cf. Notation 13, p. 1088) we find that this is equivalent to saying that $\mathrm{th}_n(t_r \otimes_c i)$ has the LLP with respect to each weak complicial set A . Now, it is easily seen that th_n preserves the Gray tensor \otimes , in the sense that we literally have $\mathrm{th}_n(X \otimes Y) = \mathrm{th}_n(X) \otimes \mathrm{th}_n(Y)$, so it follows that $\mathrm{th}_n(t_r \otimes_c i) \cong \mathrm{th}_n(t_r) \otimes_c \mathrm{th}_n(i)$ (in Strat^2). However, since $r > n$ we find that $\mathrm{th}_n(t_r)$ is actually the identity on the stratified set $\mathrm{th}_n(\Delta[r])$, from which it follows that its corner tensor with any inclusion is an isomorphism and thereby demonstrate that $\mathrm{th}_n(t_r \otimes_c i)$ has the LLP with respect to any stratified set. This certainly establishes that the condition of the last paragraph holds for J_n and thus that it satisfies Definition 91. Consequently it gives rise to a Quillen model structure whose fibrant objects we may show to be the n -trivial weak complicial sets (by applying the subsequent lemma).

Lemma 105. *If $p: A \rightarrow B$ is a J -complicial fibration of J -weak complicial sets then it is a completely J -complicial fibration. In particular, it follows that the fibrant objects of the J -complicial model structure are precisely the J -weak complicial sets.*

Proof (following an argument due to Quillen [15]). Suppose that the inclusion $e: U \hookrightarrow V$ is a J -complicial cofibration and consider the following defining diagram for the corner tensor $\mathrm{hom}^c(e, p)$:

$$\begin{array}{ccccc}
 \mathrm{hom}(V, A) & & \xrightarrow{\mathrm{hom}(e, A)} & & \mathrm{hom}(U, A) \\
 \searrow \mathrm{hom}^c(e, p) & & & & \downarrow \mathrm{hom}(U, p) \\
 & \mathrm{hom}(V, B) \times_{\mathrm{hom}(U, B)} \mathrm{hom}(U, A) & \xrightarrow{q} & & \mathrm{hom}(U, A) \\
 \searrow \mathrm{hom}(V, p) & \downarrow & \lrcorner & & \downarrow \mathrm{hom}(U, p) \\
 & \mathrm{hom}(V, B) & \xrightarrow{\mathrm{hom}(e, B)} & & \mathrm{hom}(V, B)
 \end{array}$$

in which each object is a J -weak complicial set by Observation 96, which applies here since A and B are both J -weak complicial sets. Applying Lemma 92 we may show that the maps $\mathrm{hom}(e, A)$ and $\mathrm{hom}(e, B)$ are trivial fibrations, from which it follows that the pullback q of the latter is also a trivial fibration and thus that these are all homotopy equivalences by Lemma 88. Now we can use the 2-of-3 property to demonstrate that $\mathrm{hom}^c(e, p)$ is also a homotopy equivalence and furthermore show that it is a complicial fibration, by applying Theorem 75 to the inclusion e , thus allowing us to infer that it is a trivial fibration by applying Lemma 88.

Finally, the fact that trivial fibrations have the RLP with respect to the (unique) inclusion $!: \emptyset \hookrightarrow \Delta[0]$ implies that they are all surjective on 0-simplices. Applying this to $\mathrm{hom}^c(e, p)$ we immediately see that p has the RLP with respect to e as required. \square

Corollary 106. *An inclusion $e: U \hookrightarrow V$ is a J -complicial cofibration if and only if it has the LLP with respect to each J -complicial fibration $p: A \rightarrow B$ of J -weak complicial sets.*

Proof. The “only if” part was established in the last lemma. For the “if” part observe that if A is a J -weak complicial set, and $i : X \hookrightarrow Y$ is any inclusion then we may apply Observation 96 to show that the stratified map $\text{hom}(i, A) : \text{hom}(Y, A) \rightarrow \text{hom}(X, A)$ is a J -complicial fibration of J -weak complicial sets. So by the assumption of the statement we know that e has the LLP with respect to $\text{hom}(i, A)$ and may apply Observation 122 under the adjunction $- \otimes_c i \dashv \text{hom}^c(i, *)$ to show that this is equivalent to saying that $e \otimes_c i$ has the LLP with respect to A . It follows that e satisfies clause (iii) of Lemma 92 by which we may infer that it is a J -complicial cofibration as postulated. \square

Observation 107. The J -complicial model structure is *monoidal* with respect to the Gray tensor product \otimes . This amounts to showing that if $e : U \hookrightarrow V$ is a J -complicial cofibration and $i : X \hookrightarrow Y$ is any inclusion then their corner tensor $e \otimes_c i$ is also a J -complicial cofibration.

We use clause (iii) of Lemma 92 to demonstrate this result, so suppose that $j : S \hookrightarrow T$ is any other inclusion of stratified sets and consider $(e \otimes_c i) \otimes_c j \cong e \otimes_c (i \otimes_c j)$ (in Strat^2). We know that $i \otimes_c j$ is an inclusion and that e is a J -complicial cofibration, so we may apply Lemma 92(iii) to show that $e \otimes_c (i \otimes_c j)$ has the LLP with respect to each J -weak complicial set. It follows that the isomorphic map $(e \otimes_c i) \otimes_c j$ also has this property for each j , which fact allows us to apply the same characterisation in the reverse direction to show that $e \otimes_c i$ is a J -complicial cofibration as required.

6.4. Monoidality of the complicial model structure

In this subsection we round out the results presented in Lemma 25 and Corollary 76 by extending them to encompass complicial cofibrations and to the lax Gray tensor product. As a result we establish that the complicial model structure makes Strat into a *monoidal model category* with respect to either one of the Gray tensors \otimes or \otimes . Indeed, it is trivially the case that it is also a monoidal model category with respect to the join \oplus , a result we leave to the reader to verify.

Corollary 108 (of Lemma 25). *For each $n \in \mathbb{N}$ the n -trivialisation functor th_n (cf. Notation 13, p. 1088) preserves complicial cofibrations. It follows that if $e : U \hookrightarrow V$ is a complicial cofibration then so is the associated inclusion $\text{th}_n(U) \vee_U V \hookrightarrow \text{th}_n(V)$.*

Proof. Corollary 106 tells us that $\text{th}_n(e) : \text{th}_n(U) \hookrightarrow \text{th}_n(V)$ is a complicial cofibration if and only if it has the LLP with respect to complicial fibrations of complicial sets $p : A \rightarrow B$. Taking duals under the adjunction $\text{th}_n \dashv \text{sp}_n$ (cf. Notation 13, p. 1088) we find that this is the case iff $e : U \hookrightarrow V$ has the LLP with respect to $\text{sp}_n(p) : \text{sp}_n(A) \rightarrow \text{sp}_n(B)$. This latter fact follows directly from Lemma 25 which tells us that sp_n preserves weak complicial sets and complicial fibrations.

For the second part of the statement, assume without loss of generality that e is actually a stratified subset inclusion $U \xrightarrow{\subseteq_s} V$ and that, consequently, the pushout $\text{th}_n(U) \vee_U V$ is actually the union $\text{th}_n(U) \cup V$ of subsets in $\text{th}_n(V)$. Now consider the following diagram

$$\begin{array}{ccccc}
 U & \xrightarrow{\subseteq_s} & V & & \\
 \downarrow \subseteq_e & & \downarrow \subseteq_e & & \\
 \text{th}_n(U) & \xrightarrow{\subseteq_s} & \text{th}_n(U) \cup V & \xrightarrow{\subseteq_e} & \text{th}_n(V)
 \end{array}$$

in which it is easily checked that the square is a pushout as marked. By assumption the upper horizontal here is a complicial cofibration, so it follows that the lower horizontal in this square is also a complicial cofibration. Furthermore, in the last paragraph we demonstrated that the composite of the lower horizontals is a complicial cofibration. So we may apply the 2-of-3 property to show that the right hand lower horizontal is also a complicial cofibration as required. \square

Theorem 109. *If $e : U \hookrightarrow V$ is a complicial cofibration and $i : X \hookrightarrow Y$ is any inclusion of stratified sets then their corner tensors $e \otimes_c i$ and $e \boxtimes_c i$ are complicial cofibrations. Dually it is the case that $i \otimes_c e$ and $i \boxtimes_c e$ are also complicial cofibrations.*

Proof. We avoided proving this kind of result in Section 5 because at the time we had no completely satisfactory way of relating the properties of corner tensors with respect to the lax Gray tensors \otimes and \boxtimes . However, the complicial cofibrations of our model structure provide a solution to this problem and allow us to easily prove properties of \otimes using the colimit preservation properties of \boxtimes . To that end consider the following commutative square

$$\begin{array}{ccc} (U \boxtimes Y) \vee_{U \boxtimes X} (V \boxtimes X) & \xrightarrow{e \boxtimes_c i} & V \boxtimes Y \\ \downarrow \subseteq_e & & \downarrow \subseteq_e \\ (U \otimes Y) \vee_{U \otimes X} (V \otimes X) & \xrightarrow{e \otimes_c i} & V \otimes Y \end{array}$$

in which the right hand vertical is an anodyne extension by Lemma 139 of [24] (cf. Observation 65, p. 1116), as is the left hand vertical since it is constructed as a pushout of such comparison maps. Applying Observation 97 we see that these are both weak equivalences and therefore that we can apply the 2-of-3 property to show that the upper horizontal $e \boxtimes_c i$ is a complicial cofibration if and only if the lower horizontal $e \otimes_c i$ is a complicial cofibration.

Applying this observation to Lemma 72 we find that the corner tensor \boxtimes_c of an elementary left or inner anodyne extension with a boundary or thin simplex inclusion is a complicial cofibration. However, the class of complicial cofibrations is closed under pushout and transfinite composition (cf. Notation 101, p. 1135) so we may apply Lemma 121 to the tensor \boxtimes , which does preserve colimits in each variable, to show that if e is a left anodyne extension and i is any inclusion then their corner tensor $e \boxtimes_c i$ is a complicial cofibration. In particular, we may now apply Corollary 105 to show that $e \boxtimes_c i$ has the LLP with respect to each complicial fibration $p : A \twoheadrightarrow B$ of weak complicial sets. Dually, applying Observation 122 to the adjunction $-\boxtimes_c i \dashv \text{lax}_r^c(i, *)$ we see that $\text{lax}_r^c(i, p)$ is a left complicial fibration.

Finally, to extend the work of the last paragraph to all complicial cofibrations e observe that the result given in its last sentence now allows us to apply the argument of Theorem 75 with respect to \boxtimes_c and lax_r^c . This demonstrates that $\text{lax}_r(X, A)$ is a weak complicial set whenever A is and that for each inclusion i and complicial fibration p of weak complicial sets the corner closure $\text{lax}_r^c(i, p)$ is a completely complicial fibration, and thus enjoys the RLP with respect to any complicial cofibration e . Dualising that result using Observation 122 and the adjunction $-\boxtimes_c i \dashv \text{lax}_r^c(i, *)$ we find that $e \boxtimes_c i$ has the LLP with respect to any such p . Consequently we may apply Corollary 106 to show that $e \boxtimes_c i$ is a complicial cofibration, and thus that $e \otimes_c i$ is also a complicial cofibration by the observation of the first paragraph of this proof. \square

6.5. A model structure for Joyal's quasi-categories

Finally, we derive model category structures on Strat and Simp whose fibrant objects are Joyal's quasi-categories. The structure we obtain on Simp was originally constructed by Joyal, although at the time of writing none of his published papers contain the detail of his construction. We provide the following construction as an independent verification of his work and in order to provide us with a presentation of this model structure which we shall apply in forthcoming work.

Definition 110. Define a set of inclusions

$$J_q \stackrel{\text{def}}{=} J_1 \cup \{\tilde{E}_2 \xrightarrow{\subseteq_e} E_2\}$$

where J_1 is the set defined in Example 104 whose fibrant objects are 1-trivial weak complicial sets.

Definition 111. Consider the following set of stratified inclusions:

$$Q \stackrel{\text{def}}{=} \{\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_t \mid n = 2, 3, 4, \dots\} \cup \{\tilde{E}_2 \xrightarrow{\subseteq_e} E_2, E_1^- \xrightarrow{\subseteq_r} E_2\}.$$

Of course we know that a stratified set X has the RLP with respect to the given thin simplex inclusions iff it is 1-trivial. Furthermore, X has the RLP with respect to the remaining inclusions in Q iff every 1-simplex with an equivalence inverse is thin (RLP w.r.t. $\tilde{E}_2 \xrightarrow{\subseteq_e} E_2$) and every thin 1-simplex has an equivalence inverse (RLP w.r.t. $E_1^- \xrightarrow{\subseteq_r} E_2$). It follows that the stratification of a Q -fibrant stratified set is completely determined by the structure of its underlying simplicial set.

Now, if X is a simplicial set then let X^e denote the entire superset (cf. Definition 9, p. 1087) of $\text{th}_1(X)$ constructed by making thin those 1-simplices $x \in X$ for which there is a simplicial map $\hat{x}: \tilde{E}_2 \rightarrow X$ with $\hat{x}(e_1^-) = x$. It is easily seen that X^e is Q -fibrant for any simplicial set X , so it follows from the last paragraph that each simplicial set carries a unique Q -fibrant stratification.

Furthermore, it is clear that we may construct the entire inclusion $X \xrightarrow{\subseteq_e} X^e$ as a pushout of a coproduct of copies of thin simplex inclusions and the inclusion $\tilde{E}_2 \xrightarrow{\subseteq_e} E_2$, from which it follows that it is both a relative Q -cell complex and a J_q -anodyne extension.

Clearly any simplicial map $f: X \rightarrow Y$ is the underlying map of a stratified map $f^e: X^e \rightarrow Y^e$, in other words the stratification operation $(-)^e$ provides us with a fully-faithful functor from Simp to Strat which makes the family of entire inclusions $X \xrightarrow{\subseteq_e} X^e$ into a natural transformation. So the stratification operation $(-)^e$ provides us with an equivalence between Simp and the full sub-category of Q -fibrant stratified sets in Strat.

Observation 112 (*Quasi-categories and J_q -weak complicial sets*). Returning to Example 57 we see that the functor $(-)^e$ generalises the canonical stratification discussed there. That example tells us that a simplicial set A is a quasi-category iff A^e is a weak complicial set. We know however that A^e has the RLP with respect to the set Q of the last definition and that $J_q \subseteq J_c \cup Q$, so it follows that A is a quasi-category iff A^e is a J_q -weak complicial set.

We do not yet know that every J_q -weak complicial set is of the form A^e for some quasi-category A , however that result would follow from the observations made in the course of the last definition as soon as we demonstrate that every J_q -weak complicial set is Q -fibrant. To that end observe that every element of Q *except* for the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$ is in J_q and that it is possible to show that this latter inclusion is also a (J_q) -complicial cofibration, as we do in the next two paragraphs. Consequently, we know that any complete J_q -complicial fibration is also a Q -fibration, which result immediately specialises to the one outlined in the first sentence above.

So to complete the proof outlined in the last paragraph, we start by considering the inclusion

$E_0^- \xrightarrow{\subseteq_r} E_2$ and its right inverse $q: E_2 \twoheadrightarrow E_0^-$ (the unique map into $E_0^- \cong \Delta[0]$). Consider now the order preserving map $h: \mathbb{I} \times [1] \rightarrow \mathbb{I}$ where \mathbb{I} is the two-point chaotic category of Definition 41 and h is defined by $h(p, 0) = p$ and $h(p, 1) = -$. Applying the categorical nerve construction of Observation 4 we obtain a simplicial map $h: \tilde{E} \times \Delta[1] \rightarrow \tilde{E}$ which we may stratify and restrict to give a simple homotopy $h: E_2 \otimes \Delta[1]_t \rightarrow E_2$ from the identity on E_2 to the composite $E_2 \xrightarrow{q} E_0^- \xrightarrow{\subseteq_r} E_2$. This demonstrates that q and $E_0^- \xrightarrow{\subseteq_r} E_2$ are mutual homotopy inverses and thus that they are both weak equivalences, by Observation 94.

Now we may factor the inclusion of the last paragraph as the composite of the inclusions $E_0^- \xrightarrow{\subseteq_r} E_1^-$ and $E_1^- \xrightarrow{\subseteq_r} E_2$. However the first of these is isomorphic to the left horn inclusion $\Delta^0[1] \xrightarrow{\subseteq_r} \Delta^0[1]$ and is thus a weak equivalence. It follows therefore, by the result of the last paragraph and an application of the two of three property, that the latter of these is also a weak equivalence. In other words, we have established that both of the inclusions $E_0^- \xrightarrow{\subseteq_r} E_2$ and $E_1^- \xrightarrow{\subseteq_r} E_2$ are complicial cofibrations as required.

Lemma 113. *There exists a cofibrantly generated Quillen model structure on Strat whose fibrant objects are precisely the canonically stratified quasi-categories (cf. Example 57, p. 1113).*

Proof. Given the work of the last observation, it is clear that if we may apply Theorem 100 to the set of inclusions J_q then the resulting Quillen model structure would satisfy the condition postulated with respect to quasi-categories. So following Observation 103 we must show that

$\tilde{E}_2 \xrightarrow{\subseteq_e} E_2$ is a J_q -weak equivalence and, as discussed there, we may do so by demonstrating that clause (iii) of Lemma 92 holds for each inclusion whose domain and codomain are J_1 -weak complicial sets. So assume without loss of generality that i is a subset inclusion $X \xrightarrow{\subseteq_s} Y$ and that X and Y are J_1 -weak complicial sets and consider the corner tensor:

$$(E_2 \otimes X) \cup (\tilde{E}_2 \otimes Y) \xrightarrow{\subseteq_e} E_2 \otimes Y. \quad (14)$$

Now let U denote the underlying simplicial set common to the domain and codomain of this inclusion. Since the entire inclusion $U \xrightarrow{\subseteq_e} U^e$ of Definition 111 is a J_q -anodyne extension, it is clear that we may demonstrate that the inclusion above has the LLP with respect to each J_q -weak complicial set simply by showing that its codomain $E_2 \otimes Y$ is an entire subset of U^e . However this latter set is 1-trivial so it is enough to check that each thin 1-simplex of $E_2 \otimes Y$

is also thin in U^e . To that end suppose that (e, y) is a thin 1-simplex in $E_2 \otimes Y$, and observe that the thin $y \in Y$ gives rise to a corresponding Yoneda map $\lceil y \rceil: E_1^- \cong \Delta[1]_t \rightarrow Y$ which we may lift along the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$, using the assumption that Y is a (J_1) -weak complicial set and Corollary 54, to give a map $\hat{y}: \tilde{E}_2 \rightarrow Y$ with $\hat{y}(e_1^-) = y$. Similarly, the identity map on E_2 , the dual map $\neg: E_2 \rightarrow E_2$ of Observation 44 and the maps which carry the whole of E_2 to the 0-simplex $-$ or $+$ provide maps which carry the simplex e_1^- to each one of the 1-simplices in E_2 , so we may adopt a corresponding notation $\hat{e}: \tilde{E}_2 \rightarrow \tilde{E}_2$ for the stratified map with $\hat{e}(e_1^-) = e$. It follows, therefore, that (e, y) is the image of the simplex e_1^- under the induced map $(\hat{e}, \hat{y}): \tilde{E}_2 \rightarrow \tilde{E}_2 \otimes Y \subseteq_e U$ thereby demonstrating that it is thin in U^e as required. \square

Corollary 114. *There exists a cofibrantly generated Quillen model structure on Simp whose:*

- *cofibrations are the simplicial inclusions,*
- *weak equivalences are those maps in Simp which are J_q -weak equivalences in Strat (under the minimal stratification), and*
- *fibrations are those $p: A \rightarrow B$ in Simp for which $p^e: A^e \rightarrow B^e$ is a completely J_q -complicial fibration in Strat.*

In particular, the fibrant objects in this model category are the quasi-categories.

Proof. We construct a Quillen model structure on Simp by restricting the J_q -model structure of Strat along the fully-faithful functor $(-)^e: \text{Simp} \rightarrow \text{Strat}$ of Definition 111. In other words, we define classes of cofibrations, fibrations and weak equivalences by saying that a simplicial map $f: X \rightarrow Y$ is a cofibration (resp. fibration or weak equivalence) if and only if the stratified map $f^e: X^e \rightarrow Y^e$ is a J_q -complicial cofibration (resp. complete J_q -complicial fibration or J_q -weak equivalence). Now we simply verify Quillen's Axioms M1 to M5 (see Definition 7.1.3 of [10] for instance) for this choice. Axiom M1 (limits and colimits) is immediate for the presheaf category Simp whereas Axioms M2 to M4 (2-of-3, retract and lifting) are all immediate consequences of the corresponding axioms for the J_q -complicial model structure and the fact that $(-)^e: \text{Simp} \rightarrow \text{Strat}$ is a fully-faithful functor.

That simply leaves us to verify Axiom M5 (factorisation), which postulates that we may factor each simplicial map $f: X \rightarrow Y$ as a composite $f = p \circ i$ wherein p is a fibration (resp. trivial fibration) and i is a trivial cofibration (resp. cofibration). However, we know that we may factor the stratified map $f^e: X^e \rightarrow Y^e$ as a fibration, trivial cofibration (resp. trivial fibration, cofibration)

composite $X^e \xrightarrow{i} W \xrightarrow{p} Y^e$ in the J_q -complicial model structure and that this gives rise to an appropriate factorisation in Simp under the proposed model structure if and only if the stratified set W is of the form Z^e . Now we know, from the comments in Definition 111, that this latter condition holds if and only if W is Q -fibrant. Furthermore the same passage tells us that Y^e is Q -fibrant and Observation 112 demonstrates that the (trivial) fibration p is a Q -fibration, from which facts we may infer that W is also Q -fibrant as suggested.

In the Quillen model structure we have just constructed it is clear that a simplicial map is a cofibration iff it is a simplicial inclusion. Furthermore we know, from Definition 111, that the

inclusion $X \xrightarrow{\subseteq_e} X^e$ is a J_q -anodyne extension for each simplicial set X . So if $w : X \rightarrow Y$ is a simplicial map then we know that the horizontal inclusions in the square

$$\begin{array}{ccc} X & \xrightarrow{\subseteq_e} & X^e \\ w \downarrow & & \downarrow w^e \\ Y & \xrightarrow[\subseteq_e]{} & Y^e \end{array}$$

are J_q -weak equivalences and thus that we may apply the 2-of-3 property to show that w^e is a J_q -weak equivalence iff w is such in Strat. In other words, we find that w is a weak equivalence in the Quillen model structure derived in the last two paragraphs iff it is a J_q -weak equivalence as a minimally stratified map as postulated in the statement. \square

Notation 115. We call the Quillen model structure derived in the last corollary the *quasi-categorical model structure* and use the terms *quasi-cofibration* for its trivial cofibrations and *complete quasi-fibration* for its fibrations.

Definition 116. We say that a map $p : A \rightarrow B$ in Simp is an *inner quasi-fibration* if it has the RLP with respect to the simplicial inclusions

$$\{ \Lambda^k[n] \xrightarrow{\subseteq_s} \Delta[n] \mid n = 2, 3, \dots \wedge 0 < k < n \} \quad (15)$$

that is to say these are what Joyal calls *mid-fibrations*. We also say that p is a *quasi-fibration* if it is an inner quasi-fibration which also has the RLP with respect to the simplicial inclusion

$$\tilde{E}_0^- \xrightarrow{\subseteq_r} \tilde{E}_2. \quad (16)$$

Lemma 117. *Each of the inclusions in displays (15) and (16) of the last definition is a quasi-cofibration. It follows that every complete quasi-fibration is actually a quasi-fibration in the sense introduced there.*

Proof. We know, from the last theorem, that it is enough to show that the stratified maps obtained by applying the functor $(-)^e$ to the simplicial inclusions in the cited displays are all J_q -weak equivalences. However only the degenerate simplices of $\mathrm{th}_1(\Delta[n])$ have equivalence

inverses, so if we apply the functor $(-)^e$ to the inner horn inclusion $\Lambda^k[n] \xrightarrow{\subseteq_s} \Delta[n]$ then we obtain a stratified inclusion which may otherwise be constructed by applying $\mathrm{th}_1 : \underline{\mathrm{Strat}} \rightarrow \underline{\mathrm{Strat}}$

to the inner simplicial horn inclusion $\Lambda^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$. Applying Lemma 25 it follows that this inclusion is an inner anodyne extension and is thus also a (J_q) -weak equivalence. It is also clear that the simplicial maps $\mathrm{id} : \tilde{E}_2 \rightarrow \tilde{E}_2$ and $\neg : \tilde{E}_2 \rightarrow \tilde{E}_2$ demonstrate that the 1-simplices e_1^- and

e_1^+ are thin in $(\tilde{E}_2)^e$, so when we apply $(-)^e$ to the inclusion $\tilde{E}_0^- \xrightarrow{\subseteq_r} \tilde{E}_2$ we obtain the stratified inclusion $E_0^- \xrightarrow{\subseteq_r} E_2$ which was shown to be a (J_q) -weak equivalence in Observation 112. \square

Lemma 118. *If the simplicial map $p : A \rightarrow B$ is a quasi-fibration between quasi-categories then it is a complete quasi-fibration.*

Proof. Since A and B are quasi-categories we know, by Observation 112, that the stratified sets A^e and B^e are J_q -weak complicial sets. It follows immediately, by the comment in Definition 23, that $p^e : A^e \rightarrow B^e$ has the RLP with respect to each elementary thinness extension $\Delta^k[n]' \xrightarrow{\subseteq_e} \Delta^k[n]''$, each thin simplex inclusion $\Delta[n] \xrightarrow{\subseteq_e} \Delta[n]_r$ ($n > 1$) and the equivalence inclusion $\tilde{E}_2 \xrightarrow{\subseteq_e} E_2$. Furthermore, it is easily seen that p^e has the RLP with respect to the inner horn inclusions $\Lambda^k[n] \xrightarrow{\subseteq_r} \Delta^k[n]$ ($0 < k < n$) and the equivalence inclusion $E_0^- \xrightarrow{\subseteq_r} E_2$ iff p has the RLP with respect to their underlying simplicial maps. These are, however, the simplicial inclusions used to describe quasi-fibrations in Definition 116, so it follows from the postulated properties of p that p^e does indeed have the RLP with respect to the stratified inclusions of the last sentence.

To summarise we have shown that, under the conditions of the statement, the stratified map $p^e : X^e \rightarrow Y^e$ is an inner complicial fibration between J_q -weak complicial sets and that it also has the RLP with respect to the inclusion $E_0^- \xrightarrow{\subseteq_r} E_2$. Consequently Lemma 55 tells us that p^e is a (J_q -)complicial fibration iff it has the RLP with respect to the inclusion $E_0^- \xrightarrow{\subseteq_r} E_1^-$, a result we establish by constructing a solution to the arbitrary lifting problem (u, v) depicted in the following square:

$$\begin{array}{ccc}
 E_0^- & \xrightarrow{u} & X^e \\
 \downarrow \subseteq_r & \nearrow \subseteq_r & \downarrow p^e \\
 & E_2 & \\
 E_1^- & \xrightarrow{v} & Y^e
 \end{array}$$

(Note: The diagram above is a simplified representation of the commutative square in the image. The actual diagram shows a square with vertices E_0^- , X^e , E_1^- , and Y^e . Arrows are: $E_0^- \xrightarrow{u} X^e$, $E_1^- \xrightarrow{v} Y^e$, $E_0^- \xrightarrow{\subseteq_r} E_1^-$, and $X^e \xrightarrow{p^e} Y^e$. A triangle is formed by $E_0^- \xrightarrow{\subseteq_r} E_2$, $E_1^- \xrightarrow{\subseteq_r} E_2$, and $E_0^- \xrightarrow{\subseteq_r} E_1^-$. Arrows w and v' connect E_2 to X^e and Y^e respectively.)

Start by factoring the map v through the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$, which we may do since we know that this inclusion is a complicial cofibration (Observation 112) and that Y^e is a weak complicial set, to give the map v' . Now we have a lifting problem (u, v') from the inclusion $E_0^- \xrightarrow{\subseteq_r} E_2$ to p^e for which we may find a solution w since we know that the latter map has the RLP with respect to the former. Finally we may take the composite of w and the inclusion $E_1^- \xrightarrow{\subseteq_r} E_2$ as the required solution to our original problem.

All that remains now is to apply Lemma 105 to show that p^e is actually a complete J_q -complicial fibration and thus that, by definition, p is a complete quasi-fibration as postulated. \square

Appendix A. Some categorical homotopy theory

We recollect here a few basic results of categorical homotopy theory upon which we rely in the body.

Definition 119 (*Categories of morphisms*). If \mathcal{C} is a category then its *category of morphisms* \mathcal{C}^2 is defined to be the category of functors from the ordinal $2 = \{0 < 1\}$ to \mathcal{C} . Its objects are simply morphisms $f : A \rightarrow B$ of \mathcal{C} and its arrows from f to another morphism $g : C \rightarrow D$ are pairs of arrows (u, v) of \mathcal{C} making the obvious naturality square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{v} & D \end{array}$$

commute and which are thus called *squares*. In the context of Quillen model categories, we often think of the arrows of \mathcal{C}^2 as being *lifting problems* in \mathcal{C} .

Recall 120 (*The corner tensor and its closures*). It is common in the theory of Quillen model categories to consider *pushout corner maps* so we recall the basic concepts and notation here in a suitably generalised setting.

Let \mathcal{C} , \mathcal{D} and \mathcal{E} be categories which are cocomplete and let $\odot : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ be a bifunctor (tensor) which preserves these colimits in each variable. Now suppose that $f : C \rightarrow C'$ is an arrow of \mathcal{C} and $g : D \rightarrow D'$ is an arrow of \mathcal{D} then we may consider the commutative square

$$\begin{array}{ccc} C \odot D & \xrightarrow{f \odot D} & C' \odot D \\ C \odot g \downarrow & & \downarrow C' \odot g \\ C \odot D' & \xrightarrow{f \odot D'} & C' \odot D' \end{array}$$

which induces a unique map usually denoted $f \odot_C g$ from the pushout $(C' \odot D) \vee_{C \odot D} (C \odot D')$ of the upper horizontal and left-hand vertical maps in this square to its lower right vertex $C' \odot D'$ making the usual triangles commute. This map is often called the *corner tensor* (or sometimes the *Liebnitz tensor*) of f and g and, for instance, it plays a central role in Quillen's theory of simplicial model categories [15] (for a suitable \odot). This construction provides us with a naturally defined bifunctor $\odot_C : \mathcal{C}^2 \times \mathcal{D}^2 \rightarrow \mathcal{E}^2$ which again preserves colimits in each variable.

Generally \odot will be closed in each variable, meaning that for each $C \in \mathcal{C}$ the functor $C \odot - : \mathcal{D} \rightarrow \mathcal{E}$ has a right adjoint $\text{cls}_l(C, *) : \mathcal{E} \rightarrow \mathcal{D}$ (and dually for objects $D \in \mathcal{D}$). In this case, the corner tensor \odot_C is also closed in each variable with the (left) *corner closure* $\text{cls}_l^C(f, h)$ of morphisms $f : C \rightarrow C' \in \mathcal{C}$ and $h : E \rightarrow E' \in \mathcal{E}$ being the unique map induced by the commutative square

$$\begin{array}{ccc} \text{cls}_l(C', E) & \xrightarrow{\text{cls}_l(f, E)} & \text{cls}_l(C, E) \\ \text{cls}_l(C', h) \downarrow & & \downarrow \text{cls}_l(C, h) \\ \text{cls}_l(C', E') & \xrightarrow{\text{cls}_l(f, E')} & \text{cls}_l(C, E') \end{array}$$

from its upper left vertex to the pullback $\text{cls}_l(C, E) \times_{\text{cls}_l(C, E')} \text{cls}_l(C', E')$ of its right vertical and lower horizontal maps.

Most importantly, the corner tensor is well behaved with respect to cellular completions (cf. Notation 20, p. 1091) of sets of morphisms:

Lemma 121. *Let I and J be sets of morphisms of \mathcal{C} and \mathcal{D} and let \mathcal{K} be a class of morphisms of \mathcal{E} which is closed under pushout and transfinite composition. In particular, we may also take \mathcal{K} to be $\text{cell}(\mathcal{K})$ for some set of morphisms \mathcal{K} in \mathcal{E} .*

Suppose also that we know that whenever $i \in I$ and $j \in J$ then their corner tensor $i \odot_c j$ is in \mathcal{K} . Then whenever f is a morphism in $\text{cell}(I)$ and g is a morphism in $\text{cell}(J)$ we may infer that their corner tensor $f \odot_c g$ is in \mathcal{K} .

Proof. The proof here is entirely standard and is left to the reader. \square

Observation 122. On interpreting the arrows of \mathcal{E}^2 and \mathcal{D}^2 as lifting problems in \mathcal{E} and \mathcal{D} (respectively) it is worth observing that if we are given $f: C \rightarrow C' \in \mathcal{C}$, $g: D \rightarrow D' \in \mathcal{D}$ and $h: E \rightarrow E' \in \mathcal{E}$ then the adjunction $f \odot_c - \dashv \text{cls}_f^c(f, *)$ sets up a bijection between lifting problems

$$\begin{array}{ccc}
 (C' \odot D) \vee_{C \odot D} (C \odot D') & \xrightarrow{u} & E \\
 f \odot_c g \downarrow & \nearrow l & \downarrow h \\
 C' \odot D' & \xrightarrow{v} & E'
 \end{array}
 \qquad
 \begin{array}{ccc}
 D & \xrightarrow{u'} & \text{cls}_l(C', E) \\
 g \downarrow & \nearrow \hat{l} & \downarrow \text{cls}_f^c(f, h) \\
 D' & \xrightarrow{v'} & \text{cls}_l(C, E) \times_{\text{cls}_l(C, E')} \text{cls}_l(C', E')
 \end{array}$$

in \mathcal{E} and \mathcal{D} respectively. Furthermore, as indicated a map $l: C' \odot D' \rightarrow E$ in \mathcal{E} is a solution of the lifting problem on the left if and only if the dual map $\hat{l}: D' \rightarrow \text{cls}_l(C', E)$ in \mathcal{D} under the adjunction $C' \odot - \dashv \text{cls}_l(C', *)$ is a solution of the dual lifting problem on the right. It follows, therefore, that under the conditions of the last lemma if h is a K -fibration and f is an I -cofibration then $\text{cls}_f^c(f, p)$ is a J -fibration.

Observation 123 (*The small object argument*). Almost all constructions of Quillen model structures rely upon some version of Quillen's small object argument. For instance the proof of Jeff Smith's construction theorem rests upon a variant of this construction presented in Section III.6 of [1]. Explicitly, if J is a (small) set of morphisms of our locally presentable category \mathcal{C} then Proposition III.8 of that work allows us to construct a *weak reflection* of \mathcal{C} into its full subcategory \mathcal{C}_J of J -injective (J -fibrant) objects (called the *injectivity class associated with J*). We recall, and slightly recast, their construction here in order to extract a few of the properties of the resulting weak reflection which are not discussed explicitly in [1].

We will assume that we are given a (small) set of morphisms J in a locally presentable category \mathcal{C} and adopt the notation U_j and V_j for the domain and codomain of a morphism $j \in J$ (respectively). We also assume, by an appeal to Corollary 2.3.12 of [14], that we have chosen a regular cardinal κ for which \mathcal{C} is locally κ -presentable and for which the domains U_j and

codomains V_j of the elements of J are all κ -presentable. Now we start by constructing a pointed endo-functor (F, ϕ) on \mathcal{C} by forming a (pointwise) pushout

$$\begin{array}{ccc} \coprod_{j \in J} \mathcal{C}(U_j, -) \bullet U_j & \xrightarrow{\coprod_{j \in J} \mathcal{C}(U_j, -) \bullet j} & \coprod_{j \in J} \mathcal{C}(U_j, -) \bullet V_j \\ \downarrow \text{id}_{\mathcal{C}} & & \downarrow \text{id}_{V_j} \\ \text{id}_{\mathcal{C}} & \xrightarrow{\phi} & F \end{array}$$

in the endo-functor category $[\mathcal{C}, \mathcal{C}]$. Here we use $X \bullet W$ to denote the X -fold coproduct of W with itself and we define $\coprod_{j \in J} \mathcal{C}(U_j, X) \bullet U_j \rightarrow X$, the left hand vertical in this square, to be the map induced by the family whose component from the copy of U_j which corresponds to some $f \in \mathcal{C}(U_j, X)$ is simply the morphism f itself. Notice that each representable $\mathcal{C}(U_j, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is κ -accessible (preserves κ -filtered colimits), since U_j is κ -presentable, and that the tensor $- \bullet W$ preserves all colimits, so it follows that each functor $\mathcal{C}(U_j, -) \bullet W$ is κ -accessible. Now the full subcategory of κ -accessible endo-functors is closed under colimits, since colimits commute with colimits, so consequently F is also κ -accessible since it is a pushout of coproducts of κ -accessible functors. To complete their construction, we iterate F to obtain a transfinite chain of powers of F with

$$\begin{aligned} F^0 &\stackrel{\text{def}}{=} \text{id}_{\mathbf{Strat}}, \\ F^{\alpha^+} &\stackrel{\text{def}}{=} F \circ F^\alpha \quad \text{at successor ordinals } \alpha^+, \\ F^\gamma &\stackrel{\text{def}}{=} \text{colim}_{\alpha < \gamma} (F^\alpha) \quad \text{at limit ordinals } \gamma \end{aligned}$$

and chain maps $\phi_{\alpha, \beta} : F^\alpha \rightarrow F^\beta$ (for $\alpha \leq \beta$) determined by:

$$\begin{aligned} \phi_{\alpha, \alpha^+} &\stackrel{\text{def}}{=} \phi \circ F^\alpha : F^\alpha \rightarrow F \circ F^\alpha = F^{\alpha^+} \quad \text{for an ordinal and its successor,} \\ \phi_{\alpha, \gamma} &\stackrel{\text{def}}{=} \iota_\alpha^\gamma : F^\alpha \rightarrow \text{colim}_{\alpha < \gamma} (F^\alpha) = F^\gamma \quad \text{the canonical colimit inclusion.} \end{aligned}$$

Now using the fact that the all of the objects U_j and V_j are κ -presentable in \mathcal{C} we may apply Proposition III.8 of [1] to show that F^κ is the desired weak reflection, which is κ -accessible since the class of κ -accessible functors is closed under composition and colimits.

Of course the category of morphisms \mathcal{C}^2 is also locally κ -presentable; its limits and colimits are formed pointwise in \mathcal{C} and its κ -presentable objects are those morphisms $f : A \rightarrow B$ for which A and B are both κ -presentable in \mathcal{C} . So we may apply the weak reflection result above to the set J_s of squares in \mathcal{C}^2 of the form

$$\begin{array}{ccc} U_j & \xrightarrow{j} & V_j \\ j \downarrow & & \downarrow \text{id}_{V_j} \\ V_j & \xrightarrow{\text{id}_{V_j}} & V_j \end{array}$$

for each arrow $j: U_j \rightarrow V_j$ in J , to thereby construct a functorial and κ -accessible factorisation of any arrow $f: A \rightarrow B$ of \mathcal{C} into a composite $p \circ k$ in which $p \in \text{fib}(J)$ and $k \in \text{cell}(J)$.

Observation 124 (*Injectivity classes and accessibility*). Using the fact that the domains of the maps in J are all κ -presentable, it is easily shown that the injectivity class \mathcal{C}_J associated with J is closed in \mathcal{C} under κ -filtered colimits. Furthermore, applying Corollary III.9 of [1] and the work of Section 2.3 of [14], we may construct a regular cardinal $\nu > \kappa$ for which \mathcal{C} is locally ν -presentable and for which the weak reflection of the last observation carries each ν -presentable object $A \in \mathcal{C}$ to an object $F^\kappa(A)$ which is also ν -presentable in \mathcal{C} .

Using this property, it is easily demonstrated that if \mathcal{C}_ν is the essentially small, full subcategory of ν -presentable objects in \mathcal{C} and A is an arbitrary J -injective then the comma category $(\mathcal{C}_J \cap \mathcal{C}_\nu) \downarrow A$ is ν -filtered and cofinal in $\mathcal{C}_\nu \downarrow A$. Now since \mathcal{C} is locally ν -presentable we know that A is the colimit of the canonical diagram $D_A: \mathcal{C}_\nu \downarrow A \rightarrow \mathcal{C}$ so we may infer, from the last sentence, that A is also the colimit in \mathcal{C}_J of the restricted diagram $D_A: (\mathcal{C}_J \cap \mathcal{C}_\nu) \downarrow A \rightarrow \mathcal{C}_J$. Furthermore it is clear that $\mathcal{C}_J \cap \mathcal{C}_\nu$ is essentially small and that each of its objects is ν -presentable in \mathcal{C}_J so it follows, by definition, that \mathcal{C}_J is ν -accessible.

Of course, we may apply the result above to the locally κ -presentable category of morphisms \mathcal{C}^2 and its set J_s of squares derived from J as in the final paragraph of the last observation. Doing so we find that the class $\text{fib}(J)$ of J -fibrations is always an *accessible class of maps* in \mathcal{C} . In other words, the corresponding full subcategory of \mathcal{C}^2 whose objects are the J -fibrations is both (ν -)accessible and (κ -)accessibly embedded in \mathcal{C}^2 , simply because it may otherwise be described as the injectivity class associated with J_s .

Theorem 125 (*Jeffery Smith's theorem*). *Let \mathcal{C} be a locally presentable category, \mathcal{W} be a subclass of its morphisms and I be a small set of its morphisms. Suppose further that they satisfy the following conditions:*

- (1) \mathcal{W} is closed under retracts and has the 2-of-3 property.
- (2) $\text{fib}(I)$ is a subclass of \mathcal{W} .
- (3) The class $\text{cof}(I) \cap \mathcal{W}$ is closed under transfinite composition and under pushout.
- (4) \mathcal{W} satisfies the solution set condition at I .

*Then taking \mathcal{W} as our class of **weak equivalences**, $\text{cof}(I)$ as our class of **cofibrations** and $\text{fib}(\text{cof}(I) \cap \mathcal{W})$ as our class of **fibrations** we obtain a cofibrantly generated Quillen model structure on \mathcal{C} .*

Proof. A discussion of the technical details and a full proof of this (folkloric) result may be found in Beke's work on simplicial sheaves [2]. When we apply this theorem herein we will rely on the fact that our \mathcal{W} is an accessible class of maps, which condition then ensures that condition (4) holds for any set of maps I . \square

References

- [1] J. Adámek, J. Rosický, On injectivity in locally presentable categories, Trans. Amer. Math. Soc. 336 (2) (1993) 785–804.
- [2] T. Beke, Sheafifiable homotopy model categories, Math. Proc. Cambridge Philos. Soc. 129 (3) (2000) 447–475.
- [3] J.-M. Cordier, T. Porter, Vogt's theorem on categories of homotopy coherent diagrams, Math. Proc. Cambridge Philos. Soc. 100 (1986) 65–90.

- [4] J.M. Cordier, T. Porter, Homotopy coherent category theory, *Trans. Amer. Math. Soc.* 349 (1997) 1–54.
- [5] S. Crans, On combinatorial models for higher dimensional homotopies, PhD thesis, Universiteit Utrecht, 1995.
- [6] B. Day, On closed categories of functors, in: Reports of the Midwest Category Seminar IV, in: *Lecture Notes in Math.*, vol. 137, Springer-Verlag, 1970, pp. 1–38.
- [7] W.G. Dwyer, K. Spaliński, Homotopy theories and model categories, in: *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995, pp. 73–126.
- [8] P.J. Ehlers, T. Porter, Joins for (augmented) simplicial sets, *J. Pure Appl. Algebra* 145 (2000) 37–44.
- [9] P.J. Ehlers, T. Porter, Ordinal subdivision and special pasting in quasicategories, *Adv. Math.* 217 (2) (2008) 489–518.
- [10] P. Hirschhorn, *Model Categories and Their Localizations*, *Math. Surveys Monogr.*, vol. 99, American Mathematical Society, 2003.
- [11] A. Joyal, Quasi-categories and Kan complexes, *J. Pure Appl. Algebra* 175 (2002) 207–222.
- [12] D.M. Kan, Adjoint functors, *Trans. Amer. Math. Soc.* 87 (1958) 294–329.
- [13] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, *London Math. Soc. Lecture Note Ser.*, vol. 64, Cambridge University Press, Cambridge, 1982.
- [14] M. Makkai, R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, *Contemp. Math.*, vol. 104, American Mathematical Society, 1989.
- [15] D.G. Quillen, *Homotopical Algebra*, *Lecture Notes in Math.*, vol. 43, Springer-Verlag, 1967.
- [16] J.E. Roberts, Mathematical aspects of local cohomology, in: *Proceedings of the Colloquium on Operator Algebras and Their Application to Mathematical Physics*, Marseille, 1977.
- [17] J.E. Roberts, *Complcial sets*, handwritten manuscript, 1978.
- [18] R. Steiner, Tensor products of infinity-categories, Tech. rep., University of Glasgow, 1991.
- [19] R.H. Street, Fibrations in bicategories, *Cah. Topol. Geom. Differ. Categ.* 21 (1980) 111–120.
- [20] R.H. Street, The algebra of oriented simplexes, *J. Pure Appl. Algebra* 49 (1987) 283–335.
- [21] R.H. Street, Fillers for nerves, in: *Lecture Notes in Math.*, vol. 1348, 1988, pp. 337–341.
- [22] R.H. Street, Weak omega-categories, in: *Diagrammatic Morphisms and Applications*, in: *Contemp. Math.*, vol. 318, American Mathematical Society, 2003, pp. 207–213.
- [23] D. Verity, Weak complicial sets II, nerves of complicial Gray-categories, in: A. Davydov (Ed.), *Categories in Algebra, Geometry and Mathematical Physics (StreetFest)*, in: *Contemp. Math.*, vol. 431, American Mathematical Society, 2007, pp. 441–467.
- [24] D. Verity, *Complicial Sets, Characterising the Simplicial Nerves of Strict ω -Categories*, *Mem. Amer. Math. Soc.*, vol. 905, American Mathematical Society, 2008.
- [25] R. Vogt, Homotopy limits and colimits, *Math. Z.* 134 (1) (1973) 11–52.